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NOV 78 W V MALKUS, M C THAYER N00014-78-G-0072

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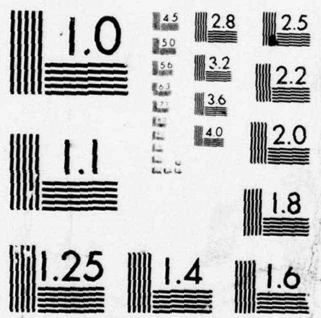

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VOLUME II

GEOPHYSICAL FLUID DYNAMICS

Great Seal

$H=X\beta\,\Omega+\overline{WT}$

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LECTURES of the FELLOWS

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(6) NOTES ON THE 1978 SUMMER STUDY PROGRAM
ON DYNAMO MODELS OF GEOMAGNETISM
IN
GEOPHYSICAL FLUID DYNAMICS
AT
THE WOODS HOLE OCEANOGRAPHIC INSTITUTION.
Volume II.

by

(10) Willem V. R. Malkus, ~~Director~~
~~and~~
Mary C. Thayer, ~~Editor~~

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EDITOR'S PREFACE

VOLUME II

This volume contains the manuscripts of research lectures by the eleven fellows of the summer program. Five of the lectures overlap significantly with the central summer theme of geomagnetism. The other six lectures cover a broad range of current G.F.D. topics from collective instability to strange attractors. Several of these research efforts are quite polished and probably will appear in journals soon. The middle half represent reports of sound progress on studies of thesis calibre. But then, a few of the lectures report on only the very first consequences of a novel idea.

These lecture reports have not been edited or reviewed in a manner appropriate for published papers. They, therefore, should be regarded as unpublished manuscripts. Readers who wish to reproduce any of the material recorded here should seek permission directly from the authors.

These two volumes represent both what we brought with us to the program and the excited first product of our scientific interactions. More sedately worded professional results invariably emerge as the year progresses. For this opportunity, we wish to thank the Woods Hole Oceanographic Institution, the Office of Naval Research, and N.A.S.A. for encouragement and financial support.

Mary C. Thayer
Willem V. R. Malkus

Roberts Keller

Top row: Spiegel, Stern, Proctor, (Weiss on T-shirt), Childress, Condi, Howard, Soward, Malkus;
Middle row: Thayer, Ierley, Hart, Holyer, Oliver, Mrs. and Mr. Mitsumoto;
Front row: Cuong, Hukuda, Freezen, Chapman, Knobloch, Whitehead.

Contents of Volume 1: Course Lectures, Seminars, and Mini-symposium Abstracts.

CONTENTS OF VOLUME 11

Lectures of the Fellows

Two papers, "Statistical Dynamics of the Lorenz Model" on p.91 and "Turbulent Diffusion of Magnetic Fields", p.133, will be found as Edgar Knobloch's Postdoctoral lectures in Volume 1.

LECTURES OF THE FELLOWS

BÉNARD CONVECTION WITH CONSTANT HEAT FLUX BOUNDARIES

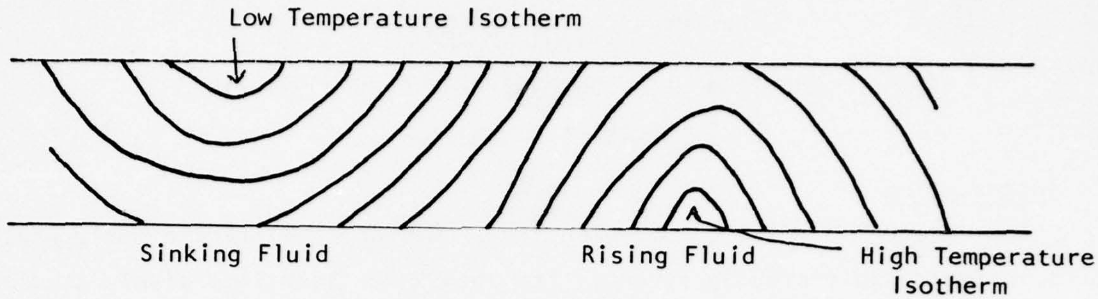
Christopher J. Chapman

(1) Introduction

The convection which occurs when fluid between two infinite horizontal planes is heated sufficiently strongly from below has been intensively studied. In most of the published analysis it is assumed that the horizontal boundaries of the fluid are perfect conductors, so that the temperature on each is constant. (The term 'constant' will invariably be used to mean 'independent of position'.) In this paper it is assumed instead that the heat flux across the boundaries is constant, so that their temperatures will depend on position once convection has begun. It would be possible in the laboratory to supply heat at a lower boundary at a rate independent of position and temperature, and one means of removing heat from the top at a constant rate would be to have cooling by evaporation. An approximation to constant heat flux is obtained by placing the fluid between two poor conductors, and a linear analysis of this situation has been given by Hurle *et al.* (1967).

(2) Effect of Fluid Motions on Temperature Distribution

Suppose that at some initial instant the fluid is motionless and the temperature varies linearly with height z, from T_0 at the bottom of the layer to T_1 at the top ($T_0 > T_1$). If we now impose a steady roll-type motion on the fluid, then in a region where the fluid is rising, the advection of the temperature profile will cause the temperature in the centre of the layer to rise; since our boundary conditions are such that the temperature gradient at the boundaries does not alter, the temperature at the boundaries must then rise and the resulting temperature profile will be approximately a linear function of z with the same gradient as before. Similarly, in regions of sinking fluid, the temperature will be its value before less an amount independent of z. The flow of heat is in the horizontal direction, and the appropriate length scale for estimating the magnitude of diffusion is the horizontal length scale of the motion, since diffusion does not alter the shape of a linear temperature profile. Thus we deduce the rather surprising fact - that a roll motion of given small velocity can produce arbitrarily large changes in the temperature if the width of the rolls is taken large enough. (This fact is important later.) The equilibrium isotherm pattern is:

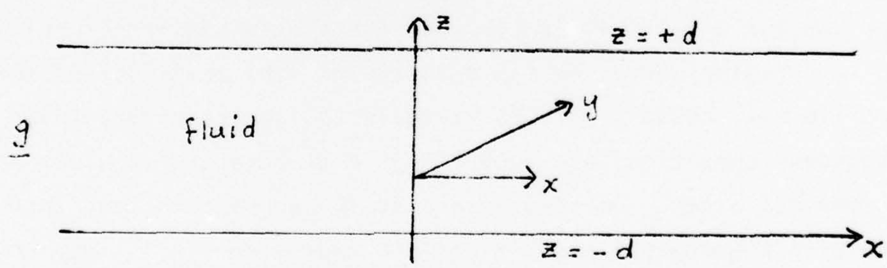


The difference in temperature between top and bottom of the layer is approximately independent of position. If the width of the rolls is very large, it will clearly take a long time for the equilibrium temperature distribution to be reached; there is thus a long initial period during which heat is slowly transferred from the regions of sinking fluid to the regions of rising fluid far away.

Note how different this is from what happens when the boundaries are held at constant temperature. In this case diffusion in the vertical direction limits the alteration of the temperature, however large the horizontal length scale.

(3) Definitions and Governing Equations

Assume that fluid of kinematic viscosity ν and thermal diffusivity χ lies between the planes z = -d and z = +d:



We take the equations describing the motion to be:

$$\begin{aligned} &\frac{\partial \underline{u}}{\partial t} + \underline{u}\cdot\nabla\underline{u} = -\frac{1}{\rho}\,\mathrm{grad}\,p - \frac{\rho}{\rho_0} g\underline{e} + \nu\nabla^2\underline{u}, \\ &\frac{\partial T}{\partial t} + \underline{u}\cdot\nabla T = \chi\nabla^2 T, \\ &\mathrm{div}\,\underline{u} = 0 \\ &\rho = \rho_0\{1-\alpha(T-T_0)\} \end{aligned} \qquad (1)$$

where $\underline{u}$ is the velocity, T is the temperature, ρ the density, p the pressure, ρ_0 the density at temperature T_0, α the coefficient of thermal expansion, and $-g\underline{e}_z$ the acceleration due to gravity. The Boussinesq approximation is

made, that the fluid can be taken to be incompressible except insofar as changes in density produce buoyancy forces. At the boundaries we assume that here is no stress in the fluid and that the temperature gradient is $-\beta\,(\beta > 0)$; so writing $\underline{u} = (u, v, w)$ we have:

$$w = 0, \quad \frac{\partial^2 w}{\partial z^2} = 0, \quad \frac{\partial T}{\partial z} = -\beta \qquad \text{on} \quad z = \pm d \tag{2}$$

The equations admit the steady conduction solution:

$$\left.\begin{aligned} \underline{u} &= \underline{u}_s = 0 \\ T &= T_s = T_0 - \beta z, \\ \rho &= \rho_s = \rho_0(1 + \alpha\beta z), \\ p &= p_s = -\rho_0 g(z + \tfrac{1}{2}\alpha\beta z^2) + \text{constant}. \end{aligned}\right\} \tag{3}$$

Define Θ, $\delta\rho$, and δp by the equations

$$\left.\begin{aligned} T &= T_s + \Theta, \\ \rho &= \rho_s + \delta\rho \\ p &= p_s + \delta p \end{aligned}\right\} \tag{4}$$

Then from (1) and (2) we obtain

$$\left.\begin{aligned} \delta\rho &= -\rho_0 \alpha \theta, \\ \frac{\partial \underline{u}}{\partial t} + \underline{u}\cdot\nabla\underline{u} &= -\frac{1}{\rho_0}\operatorname{grad}(\delta p) + g\alpha\theta\,\underline{e}_z + \nu\nabla^2\underline{u}, \\ \frac{\partial\theta}{\partial t} + \underline{u}\cdot\nabla\theta &= \beta w + \kappa\nabla^2\theta, \end{aligned}\right\} \tag{5}$$

$$w = 0, \quad \frac{\partial^2 w}{\partial z^2} = 0, \quad \frac{\partial\theta}{\partial z} = 0 \quad \text{on} \quad z = \pm d. \tag{6}$$

We shall consider only motions in the (x, z) plane and independent of y. The velocity can therefore be expressed in terms of a stream function $\Psi(x,z)$:

$$\begin{aligned} \underline{u} &= \operatorname{curl}(\Psi\underline{e}_y) \\ &= (-\Psi_z, 0, \Psi_x). \end{aligned} \tag{7}$$

After taking the curl of the momentum equation, the governing equations become:

$$\left.\begin{aligned} &\frac{\partial}{\partial t}(\nabla^2\Psi) + \frac{\partial(\Psi,\nabla^2\Psi)}{\partial(x,z)} = g\alpha\frac{\partial\theta}{\partial x} + \nu\nabla^4\Psi \\ &\frac{\partial\theta}{\partial t} + \frac{\partial(\Psi,\theta)}{\partial(x,z)} = \beta\frac{\partial\Psi}{\partial x} + \kappa\nabla^2\theta \\ &\Psi = 0, \quad \Psi_{zz} = 0, \quad \theta_z = 0 \quad \text{on} \quad z = \pm d. \end{aligned}\right\} \tag{8}$$

These equations can be made dimensionless by defining new quantities as follows:

$$\begin{aligned} &x = dx', && \theta = \beta d\theta' && R = g\alpha\beta d^4/\nu\kappa, \\ &z = dz', && \Psi = \kappa\Psi', && \sigma = \nu/\kappa, \\ &t = \frac{d^2 t}{\kappa}, \end{aligned} \tag{9}$$

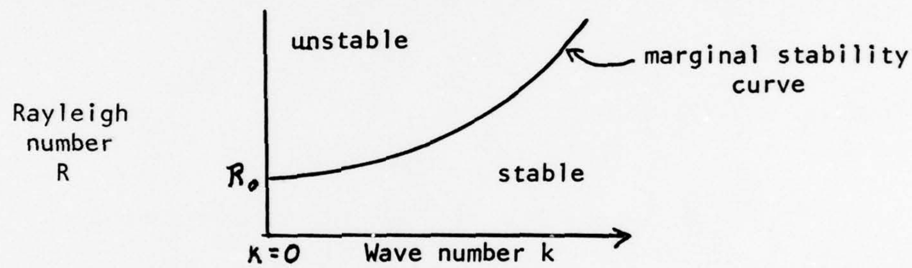
where R is the Rayleigh number and σ the Prandtl number. Equations (8) become (omitting primes):

$$\left.\begin{aligned} \frac{1}{\sigma}\left\{\frac{\partial}{\partial t}(\nabla^2\psi) + \frac{\partial(\psi,\nabla^2\psi)}{\partial(x,z)}\right\} &= R\frac{\partial\theta}{\partial x} + \nabla^4\psi \\ \frac{\partial\theta}{\partial t} + \frac{\partial(\psi,\theta)}{\partial(x,z)} &= \frac{\partial\psi}{\partial x} + \nabla^2\theta \\ \psi = \psi_{zz} = \theta_z = 0 \quad &\text{on} \quad z = \pm 1 \end{aligned}\right\} \qquad (10)$$

The problem now is: given that at some initial instant ψ and θ are small and that they evolve according to (10), what ultimately happens?

(4) The Marginal Stability Curve

The marginal stability curve for infinitesimal disturbances can be obtained from Eqs.(10), less the nonlinear terms. Hurle *et al.* (1967) have performed the calculations with the following result:



(They also show that the principle of exchange of stabilities is valid.) The critical value of R is $R_0 = 15/2$ and the critical wave number is $k = 0$; note that all Rayleigh numbers as we define them differ by a factor of 2^4 from the usual ones, because the thickness of the fluid layer is 2d not d - this simplifies some arithmetic later on. The curve has a horizontal tangent at $k = 0$, and so if the heat flux through the layer is gradually increased, then the convection first sets in with very long wave lengths. This is markedly different from the manner of onset of convection when the boundaries are maintained at constant temperature; the convection cells then first obtained have widths of the same order of magnitude as the thickness of the layer.

If the Rayleigh number is fixed at a value slightly above R_0 then on the linear theory all disturbances with wave numbers in a certain band will grow exponentially in time. Since these results can be deduced from the equations derived in the nonlinear theory (with which this paper is primarily concerned), they are discussed in subsequent sections.

(5) Some Simple Order-of-Magnitude Estimates

Suppose we have steady convection at a Rayleigh number only slightly in

excess of R_0, so that velocities are small. Let the typical cell width be L, where $L >> d$ by the results of section (4). Then from (5) and the equation div $\underline{u} = 0$, we obtain the following order of magnitude equations, in dimensional quantities:

$$\left.\begin{aligned} \frac{\kappa\theta}{L^2} &\sim w\frac{\Delta T}{d}, && (11)\\ \frac{u}{L} &\sim \frac{w}{d}, && (12)\\ \frac{\partial}{\partial x}\left(\frac{\delta p}{\rho_0}\right) &\sim \frac{\nu u}{d^2} && (13)\\ \frac{\partial}{\partial z}\left(\frac{\delta p}{\rho_0}\right) &\sim \frac{\nu w}{d^2} + g\alpha\theta, && (14)\end{aligned}\right\}$$

where ΔT is the difference in temperature between the top and bottom of the layer. This difference in temperature was shown in section (2) to be approximately constant. In (11) we have used the result of section (2) that to leading order, diffusion acts in the horizontal direction. Eliminating pressure from (13) and (14)

$$\frac{\nu w}{d^2} + g\alpha\theta \sim \frac{L}{d}\cdot\frac{\nu u}{d^2} \tag{15}$$

$$\sim \frac{L^2}{d^2}\,\frac{\nu w}{d^2} \tag{16}$$

The first term of the left of (16) is negligible compared with the term on the right since $L >> d$; thus the main balance of forces in the horizontal direction is between the pressure gradient and viscosity, while in the vertical direction it is between the pressure gradient and buoyancy. Combining the equation

$$\frac{L^2}{d^2}\,\frac{\nu w}{d^2} \sim g\alpha\theta \tag{17}$$

with (11), we see that L^2 cancels out to give

$$\frac{\alpha g\Delta T d^3}{\nu\kappa} \sim 1. \tag{18}$$

Thus, as expected, the Rayleigh number is order one. Equation (11) can be written in the form

$$\frac{\theta}{\Delta T} \sim \frac{wd}{\kappa}\,\frac{L^2}{d^2}, \tag{19}$$

which becomes, in the dimensionless variables of section (3),

$$w \sim \left(\frac{d}{L}\right)^2\theta. \tag{20}$$

The above considerations guide us to a system of scaling for a perturbation analysis of the dimensionless equation (10). Let the Rayleigh number be

$$R = R_0 + \epsilon^2, \tag{21}$$

where $0 < \epsilon << 1$. Equation (21) defines ϵ. From the shape of the marginal stability curve of section (4), the unstable wave numbers will be in a band

of thickness of order ϵ . We therefore scale the variable x by making the transformation

$$\frac{\partial}{\partial x} \longrightarrow \epsilon \frac{\partial}{\partial x}, \tag{22}$$

A nondimensional wave number of order ϵ corresponds to a dimensional wavelength of order d/ϵ . which is L as defined above. Hence $\epsilon \sim d/L$. We expect ϵ to be proportional to the amplitude of the convection as represented by the horizontal velocity component. (Compare Malkus and Veronis (1956)). Therefore we make the transformation

$$\psi \longrightarrow \epsilon \psi. \tag{23}$$

From (20), using $w = \psi_x$ and $\epsilon \sim d/L$, it is seen that θ must be order one. It turns out that the appropriate scaling for t is obtained by making the transformation

$$\frac{\partial}{\partial t} \longrightarrow \epsilon^4 \frac{\partial}{\partial t}. \tag{24}$$

The new $\frac{\partial}{\partial x}$, $\frac{\partial}{\partial t}$, ψ , and θ are thus order one.

Using the notation $D = \frac{d}{dz}$ and $\partial = \frac{\partial}{\partial x}$, transformations (22) - (24) give (10) the form:

$$\epsilon^4 \frac{\partial \theta}{\partial t} + \epsilon^2 \frac{\partial(\psi,\theta)}{\partial(x,z)} = \epsilon^2 \partial \psi + \epsilon^2 \partial^2 \theta + D^2 \theta, \tag{25}$$

$$\frac{1}{\sigma}\left\{\epsilon^6 \frac{\partial}{\partial t}(\partial^2 \psi) + \epsilon^4 \frac{\partial}{\partial t}(D^2 \psi) + \epsilon^4 \frac{\partial(\psi,\partial^2\psi)}{\partial(x,z)} + \epsilon^2 \frac{\partial(\psi, D^2\psi)}{\partial(x,z)}\right\}$$
$$= R_0 \partial \theta + \epsilon^2 \partial \theta + \epsilon^4 \partial^4 \psi + 2\epsilon^2 \partial^2 D^2 \psi + D^4 \psi. \tag{26}$$

It is assumed that σ is of order one. We now write

$$\left.\begin{aligned} \theta &= \theta_0 + \epsilon^2 \theta_2 + \epsilon^4 \theta_4 + \cdots, \\ \psi &= \psi_0 + \epsilon^2 \psi_2 + \epsilon^4 \psi_4 + \cdots, \end{aligned}\right\} \tag{27}$$

there being no terms in odd powers of ϵ , because Eqs.(25) and (26) contain no odd powers. Equations (25) and (26) can now be solved by equating powers of ϵ to give differential equations for θ_0 , ψ_0 , θ_2 , ψ_2 , ... , in that order. The boundary conditions are

$$\psi_n = D^2\psi_n = D\theta_n = 0 \quad \text{on} \quad z = \pm 1,$$

for n = 0, 2, 4, ... The calculation is given in the next section.

Of particular interest in the above scheme is the fact that the temperature perturbation is O(1) as $\epsilon \longrightarrow 0$. This is in accordance with the analysis given in section (2), and means that the equations above apply only after the initial period of convection during which a large amount of heat is transferred from regions of sinking fluid to regions of rising fluid a distance of order $1/\epsilon$ away. It would be of interest to examine this initial period more

closely, and try to match it with the later behavior. This is a possible direction for further investigation.

(6) The Perturbation Analysis

From Eqs.(25) - (27) we obtain:

Eq.(25), ϵ^0: $\quad D^2\theta_0 = 0$

Therefore by (28), $\quad \theta_0 = f(x,t)$, say.

Eq.(26), ϵ^0: $\quad D^4\psi_0 = -R_0 f_1$,

where the notation f_n stands for $\partial^n f/\partial x^n$. The boundary conditions then give

$$\psi_0 = -R_0 f_1 \left(\tfrac{1}{24}z^4 - \tfrac{1}{4}z^2 + \tfrac{5}{24}\right)$$
$$= f_1 P(z), \text{ say,}$$

where P incorporates R_0.

Eq.(25), ϵ^2:

$$D^2\theta_2 = \frac{\partial(\psi_0,\theta_0)}{\partial(x,z)} - \partial\psi_0 - \partial^2\theta_0$$
$$= -f_1^2 P_1 - f_2\{P(z)+1\},$$

where P_n denotes $d^n p/dz^n$. Now the boundary conditions $D\theta_n = 0$ on $z = \pm 1$ imply that

$$\int_{-1}^{1} D^2\theta_n \, dz = 0.$$

Thus we obtain a condition which the right-hand sides of the equations for $D^2\theta_n$ must satisfy (the secularity condition). Since p_1 is an off polynomial, and f is a bounded function of x, we obtain the value of R_0 from the equation

$$0 = \int_{-1}^{1}\{P(z)+1\}\,dz.$$

Hence $R_0 = 15/2$, in agreement with Hurle *et al.* (1967) after allowing for our different definition of the width of the layer, and so

$$p(z) = -5/16\, z^4 + 15/8\, z^2 - 25/16.$$

The equation for θ_2 can now be solved to give

$$\theta_2 = f_2\, Q(z) + f_1^2 R(z) + g(x,t),$$

where

$$Q(z) = \frac{1}{96}z^6 - \frac{5}{32}z^4 + \frac{9}{32},$$
$$R(z) = \frac{1}{16}z^5 - \frac{5}{8}z^3 + \frac{25}{16}z,$$

and g is a function which will be determined at a later stage.

Eq.(26), ϵ^2: Writing r for R_0 henceforth (to avoid confusion with R(z)), we have

$$D^4\psi_2 = \frac{1}{\sigma}\frac{\partial(\psi_0, D^2\psi_0)}{\partial(x,z)} - r\partial\theta_2 - \partial\theta_0 - 2D^2\partial^2\psi_0$$

$$= f_1 f_2 \left\{ \frac{1}{\sigma}(PP_3 - P_1 P_2) - 2rR \right\} - f_3 \left\{ 2P_2 + rQ \right\} - f_1 - rg_1 .$$

Hence $\quad \Psi_2 = f_1 f_2 \, S(z) + f_3 T(z) + f_1 u(z) + g_1 P$

where

$$T(z) = \frac{15}{192} \left\{ \left(\frac{-z^{10}}{7.8.9.10} + \frac{15 z^8}{5.6.7.8} + \frac{49 z^6}{3.4.5.6} - \frac{96 z^4}{1.2.3.4} \right) - A - B . \tfrac{1}{2}(z^2 - 1) \right\}$$

with A = value of expression in brackets (...) at z = 1,

B = value of second derivative of expression in brackets (...) at z = 1, and U (z) = $-1/24 \; (z^4 - 6z^2 + 5)$.

It turns out that the expression for S(Z) is not required; but note that it is an odd polynomial and S(1) = S(-1) = 0.

<u>Eq. (25), ϵ^4</u>

$$D^2\theta_4 = \frac{\partial \theta_0}{\partial t} + \frac{\partial(\Psi_0, \theta_2)}{\partial(x, z)} - \frac{\partial(\Psi_2, \theta_0)}{\partial(x, z)} - \partial \Psi_2 - \partial^2 \theta_2$$

$$\begin{aligned} &= \frac{\partial f}{\partial t} + f_1^2 f_2 \left\{ PR_1 - 2P_1 R - S_1 \right\} \\ &\quad + f_2^2 \left\{ PQ_1 - S - 2R \right\} \\ &\quad + f_1 f_3 \left\{ -P_1 Q - T_1 - S - 2R \right\} \\ &\quad + f_1^2 \left\{ -u_1 \right\} \\ &\quad + f_4 \left\{ -T - Q \right\} \\ &\quad + f_2 \left\{ -u \right\} - 2 f_1 g_1 P_1 - g_2 \left\{ P(z) + 1 \right\} \end{aligned}$$

The secularity condition now gives the equation for f. All odd polynomials integrate to zero, as also do the even polynomials S_1 (because S(1) = S(-1) = 0) and P(z) + 1 (by construction of P). We also have R_1 = -p. The result is therefore

$$0 = \frac{\partial f}{\partial t} - \left\{ \frac{1}{2} \int_{-1}^{1} P^2 \, dz \right\} (F_1^3)_1 - \left\{ \frac{1}{2} \int_{-1}^{1} (T + Q) \, dz \right\} f_4 - \left\{ \frac{1}{2} \int_{-1}^{1} u \, dz \right\} f_2 \tag{28}$$

Evaluation of the integrals gives

$$\frac{\partial f}{\partial t} = 1.230{,}159\,(f_1^3) - 0.787{,}157\, f_4 - \frac{2}{15} f_2 \tag{29}$$

(The detailed arithmetic has not been checked independently. Equation (29) should therefore be regarded as provisional.) This is the fundamental equation describing, to leading order, the nonlinear evolution of the system; recall that

$$\theta_0 = f(x, t), \tag{30}$$

and

$$\psi_0 = \frac{\partial f}{\partial x} P(z). \tag{31}$$

The calculation above has some interesting features. Firstly, the fact that Θ_0 is independent of Z is in accord with the plausibility argument given in section (2). Secondly, the inertia term $\frac{1}{\sigma}\frac{\partial(\Psi_0, D^2\Psi_0)}{\partial(x,z)}$ makes no contribution to the equation for f, because the term containing it integrates to zero. The equation for f therefore does not contain σ. In fact, the only terms which contribute to the nonlinearity are $\frac{\partial(\Psi_0, \Theta_0)}{\partial(x,z)}$ and $\frac{\partial(\Psi_0, \Theta_2)}{\partial(x,z)}$, which represent advection of temperature by the leading order velocity term.

(7) Proof that the Expansion Procedure is Consistent

It is necessary to prove that the procedure above, if continued, would give equations for Θ_2, Ψ_2, Θ_4, Ψ_4, ..., uniquely and without contradiction. Thus it must be demonstrated that we obtain differential equations for the functions f, g, h, ... introduced into the terms Θ_0, Θ_2, Θ_4, ... To show this, suppose that for some even $n(\geq 4)$, we have

$$\Theta_n = h(x,t) + \text{terms not involving } h(x,t).$$

Then from (26), $D^4\Psi_n = -r\frac{\partial\Theta_n}{\partial x}$ + terms of subscript less than or equal to $(n-2)$

So $\qquad \Psi_n = h_1 P(z)$ + other terms.

Now from (25), $D^2\Theta_{n+2} = \frac{\partial(\Psi_n, \Theta_0)}{\partial(x,z)} + \frac{\partial(\Psi_0, \Theta_n)}{\partial(x,z)} - \frac{\partial\Psi_n}{\partial x} - \frac{\partial^2\Theta_n}{\partial x^2}$

+ terms of subscript less than or equal to (n-2)

$= -2f_1h_1P_1 - h_2(P+1)$ + terms not involving h.

So the function $h(x,t)$ introduced into Θ_n does not appear in the secularity equation for $D^2\Theta_{n+2}$, and is determined by the secularity equation for $D^2\Theta_{n+4}$. Hence we obtain a differential equation for each of the functions introduced. Note that these differential equations (other than the one for f) contain the functions determined by the earlier differential equations in the sequence; they are linear and inhomogeneous.

(8) Linear Analysis

Equation (29), without the nonlinear terms, is

$$\frac{\partial f}{\partial t} = -b\frac{\partial^4 f}{\partial x^4} - c\frac{\partial^2 f}{\partial x^2}, \tag{32}$$

where b = 0.787157 and c = 2/15. Trying the solution

$$f(x,t) = g(t)\sin kx$$

gives

$$\frac{\partial g}{\partial t} = (-bk^4 + ck^2)\,g(t).$$

The growth rate is therefore zero when k = 0 or $\sqrt{c/b}$ (= 0.4116), and positive only for k between these values. The maximum occurs for $k = \sqrt{c/2b}$. Returning to the original units, we see that if the Rayleigh number is $R_0 + \epsilon^2$ then on the linear theory the unstable disturbances are those with

$$\text{wavelength} > 2\pi \left(b/c\right)^{1/2} \frac{d}{\epsilon} ,$$

and the disturbances of maximum growth rate are those with

$$\text{wavelength} = 2\pi \left(2b/c\right)^{1/2} \frac{d}{\epsilon} .$$

(I am not clear about the interpretation of the above results, since the linear analysis applies only to the initial period during which the order one temperature perturbations are created and after this period f is not small.)

(9) Steady Nonlinear Solutions

By an order one linear scaling of x , t , and f, the coefficients on the riaht-hand side can be given any prescribed values (or the correct sign). Here it is convenient to take the equation as

$$\frac{\partial f}{\partial t} = 2\frac{\partial}{\partial x}\left\{\left(\frac{\partial f}{\partial x}\right)^3\right\} - \frac{\partial^4 f}{\partial x^4} - \frac{\partial^2 f}{\partial x^2} . \qquad (33)$$

Later on we shall choose different values for the coefficients. (Recall that in this equation x and t are scaled by powers of ϵ .) To obtain steady solutions f(x) let g = $\frac{df}{dx}$. Then integration of (33) gives

$$\frac{d^2 g}{dx^2} + g - 2g^3 + \frac{1}{2}A = 0, \qquad (34)$$

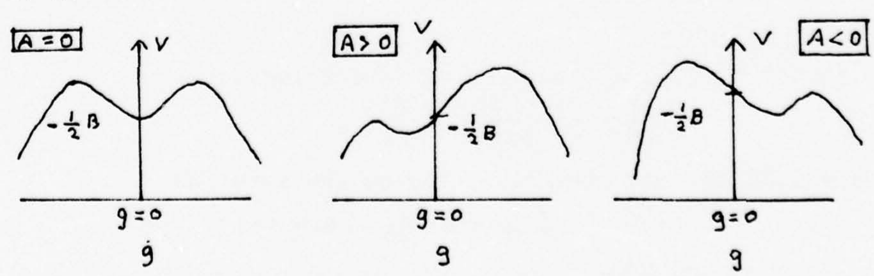
where A is a constant of integration. Hence

$$\left(\frac{dg}{dx}\right)^2 + \left(g^2 - g^4 + Ag - B\right) = 0, \qquad (35)$$

where B is another constant of integration. If we regard x as 'time' and g as 'distance' then (35) represents the motion of a particle in a potential V given by

$$V(g) = 1/2\ (-g^4 + g^2 + Ag - B),$$

and the values of g are such that V   0. The following are graphs of V for different values of A.



For an infinite plane layer, f must be bounded, and so the motion of our imaginary particle is such that the time average of its position g is zero.

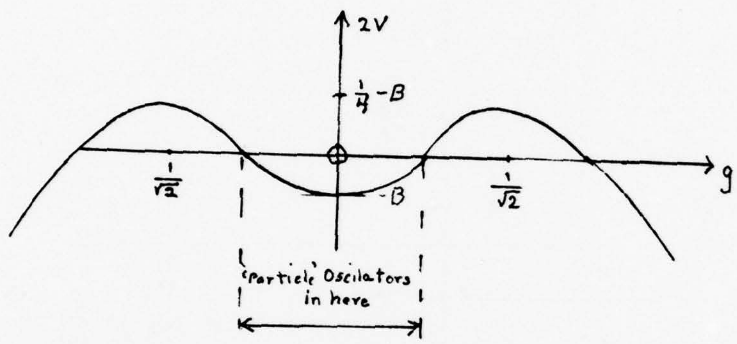
Therefore the particle must oscillate in the potential well. If $A > 0$, the oscillation is about a negative value of g, say g_0, and from the shape of the well it can be seen that the particle spends more time with $g < g_0$ than with $g > g_0$. So the time-averaged value of g is then negative. Hence we cannot have $A > 0$. Similarly, we cannot have $A < 0$. Since the bottom of the well must have a negative potential and the top a positive potential, (from (35)) we find that

$$0 \leq B \leq \tfrac{1}{4},$$

so that the graph of

$$2V = -g^4 + g^2 - B$$

is then



From (35) we obtain

$$\int dx = \int \frac{dg}{\sqrt{(g^4 - g^2 + B)}} \tag{36}$$

The integration can be performed using elliptic functions. It turns out to be convenient to define the quantities α and λ by

$$B^{-1/2} = 2 \sec \alpha, \qquad (0 \leq \alpha \leq \tfrac{1}{2}\pi)$$

$$\lambda = \tfrac{1}{2}\left(\tfrac{\pi}{2} - \alpha\right) \qquad (0 \leq \lambda \leq \tfrac{1}{4}\pi)$$

Equation (36) then gives the neat expression

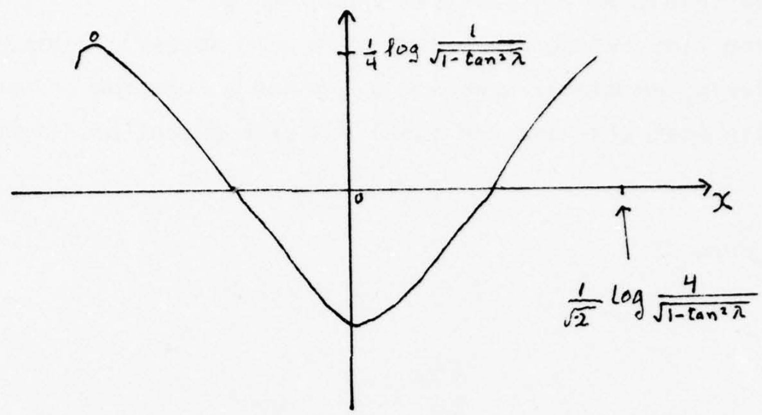
$$g(x) = (\sin \lambda)\, \mathrm{sn}\left\{(\cos \lambda)\, x,\ \tan \lambda\right\}, \tag{37}$$

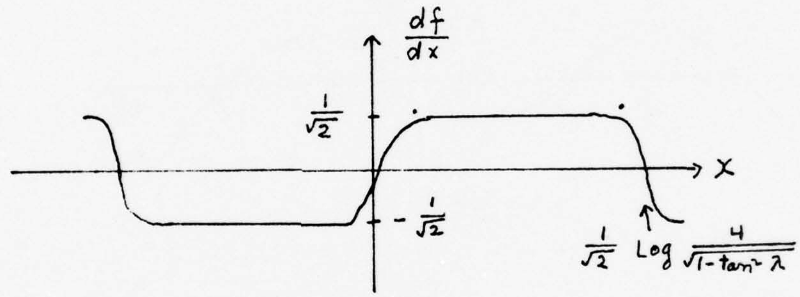
in which we ignore the arbitrary constant which can be added to x. Values of λ between 0 and $1/4\pi$ give all the steady bounded solutions of (33) on integration. The result is

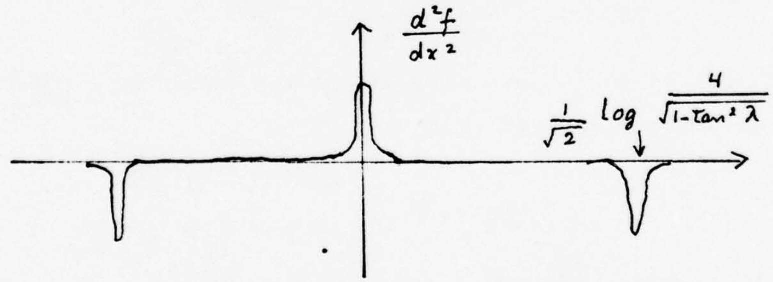
$$f(x) = \mathrm{Log}\left[\mathrm{dn}\{(\cos\lambda)x\} - (\tan\lambda)\, \mathrm{cn}\left\{(\cos\lambda)x\right\}\right] \tag{38}$$

where the constant is chosen so that the average value of f is zero, since there can be no change of the fluid's average temperature. Equations (37) and (38) show clearly the dependence of amplitude on wavelength. When λ is close to zero, $g(x)$ is approximately sinusoidal, with wavelength approximately 2π and amplitude λ.

When λ is close to $1/4\,\pi$, g has wavelength approximately $\sqrt{2}\,\log \frac{4}{\sqrt{1-\tan^2\lambda}}$ and amplitude $\frac{1}{\sqrt{2}}$. The graphs of f, df/dx, and d^2f/dx^2 are as follows (for λ near $1/4\,\pi$).







Now $\theta_0 = f$, $\Psi_0 = \frac{df}{dx}\,P(z)$, and (to leading order) $w = \frac{d^2f}{dx^2}\,P(z)$; hence these graphs show the shape of temperature and velocity distributions for steady convection at large wavelength. (To obtain the original unscaled length variable χ , the graphs should be stretched horizontally by a factor of ϵ^{-1}.)

From the symmetry of f it can be seen that there is no difference in shape between the regions of rising fluid and regions of sinking fluid. By a simple extension of the arguments given in section (2) it can be seen that if we altered the boundary conditions to have constant flux on one boundary and constant temperature on the other, then the isotherm pattern at the regions of rising fluid would be considerably different from that at regions of sinking fluid.

There would be a corresponding difference in the flow patterns, and possibly also in the horizontal length scales of up and down motions. This could be worth further study.

The stability of the steady solutions is considered in section (13).

(10) Energy Equation

It is convenient in this section to scale x, t, f, so that (29) becomes

$$\frac{\partial f}{\partial t} = (f_x^3)_x - f_{xxxx} - f_{xx} \tag{39}$$

Multiplying by f and integrating by parts gives

$$\frac{d}{dt}\int \frac{1}{2} f^2 dx = -\int f_x^4 dx - \int f_{xx}^2 dx + \int f_x^2 dx \tag{40}$$

provided that the integrated terms ff_x^3, ff_{xxx}, $f_x f_{xx}$ and ff_x all vanish. This is so if the integral is taken from $-\infty$ to $+\infty$ and $f \longrightarrow 0$ sufficiently rapidly as $x \longrightarrow \pm\infty$, or if the integral is taken over a finite range and at the boundaries

$$\left.\begin{array}{ll} \text{(i)} & f = 0 \text{ and } f_x = 0 \\ \text{or (ii)} & f = 0 \text{ and } f_{xx} = 0 \\ \text{or (iii)} & f_x = 0 \text{ and } f_{xxx} = 0 \end{array}\right\} \tag{41}$$

In this problem the natural conditions are (iii), since these correspond to zero heat flux and zero stress on the side walls, by (30) and (31).

(11) Proof that there is no Subcritical Instability

Suppose that in section (6) we put $R = R_o - \epsilon^2$ instead of $R = R_o + \epsilon^2$. Then the analysis is identical, except that in the equation for $D^2\psi^2$ we need $+\partial\Theta_o$ on the right-hand side, instead of $-\partial\Theta_o$. The expression for Ψ_2 is therefore the same as before provided that U(z) is defined as being minus the expression given. Hence in (29) we need only alter $-2/15\ f_2$ to $+2/15\ f_2$. Rescaling the variables gives

$$\frac{\partial f}{\partial t} = (f_x^3)_x - f_{xxxx} + f_{xx} \tag{42}$$

and the energy equation becomes

$$\frac{d}{dt}\int \frac{1}{2} f^2 dx = -\int f_x^4 dx - \int f_{xx}^2 dx - \int f_x^2 dx \tag{43}$$

Thus all disturbances die away and the system is stable. Hence there is no subcritical instability in the expansion scheme we have adopted.

(12) Analogue of the Nusselt Number

If we define

$$\Delta\theta(x,t) = \theta\big|_{z=-1} - \theta\big|_{z=+1},$$

then from the expressions for Θ_3 and Θ_2 in section (6), we see that

$$\Delta\Theta = -2f_1^2\epsilon^2 + O(\epsilon^4)$$

Therefore in dimensional units,

$$\frac{\Delta\theta}{\Delta T} = -f_1^2\epsilon^2 + O(\epsilon^4)$$

where $\Delta T = 2\beta d$. Thus when $R = R_o + \epsilon^2$, the effect of convection is to reduce the mean temperature difference across the layer by a factor

$$\langle(\frac{\partial f}{\partial x})^2\rangle\epsilon^2 + O(\epsilon^4),$$

where $\langle\ \rangle$ denote the mean over x.

In the Bénard problem with constant temperature boundaries, the Nusselt number is defined as the ratio of the mean heat flux passing through the layer to the heat flux which would be obtained if the heat flow were entirely by conduction. During convection the Nusselt number is greater than unity; that is, convection increases the mean heat flux for a given temperature difference between the boundaries. With constant flux boundaries, the dimensionless number of interest is the ratio of the mean temperature difference actually present to that which would be obtained if the heat flow were entirely by conduction. This number is less than unity during convection (as has just been demonstrated); that is, convection decreases the mean temperature difference for a given heat flux through the layer. When $R - R_o$ is small, this temperature difference is small; but recall that the actual temperature perturbations are large.

In much of the work that has been done on convection with constant temperature boundaries, considerations of heat flux play an important role. It would appear that all of this work will have its analogue in the constant flux problem, provided that temperature differences are the object of study. Further investigation of this may be worthwhile. For example, the reduction in mean temperature difference described above may be expected to occur for all $R > R_o$, (not just R slightly greater than R_o) and it should be possible to prove this directly from the governing equations, without using perturbation theory.

(13) Variational Method

If f(x,t) is any sufficiently smooth function of x and t, we may define V by

$$V(f,t) = \int_a^b \left(\frac{1}{4}f_x^4 + \frac{1}{2}f_{xx}^2 - \frac{1}{2}f_x^2\right)dx, \qquad (44)$$

where a and b are fixed numbers. Now let f(x,t) be any solution of (39) satisfying one of the conditions in (41) at x = a and b for all t. Then it easily

follows that

$$\frac{d}{dt} V(f,t) = -\int_a^b f_t^2 \, dx. \tag{45}$$

Thus V decreases during the motion unless the solution is steady. In the steady state, we know from the energy equation (40) that

$$0 = \int_a^b \left(-f_x^4 + f_x^2 + f_x^2\right) dx, \tag{46}$$

and so in the steady state,

$$\begin{aligned} V(f,t) &= -\frac{1}{4}\int_a^b f_x^4 \, dx \\ &= -\frac{1}{4}\int_a^b g^4 \, dx. \end{aligned} \tag{47}$$

By the Euler-Lagrange equation, the condition for stationarity of the integral in (44) (for fixed t) is

$$0 = \frac{\partial^2 g}{\partial x^2} - g^3 + g, \tag{48}$$

which is the equation for steady solutions of (39). If we specify that the layer of fluid extends from $x = a$ to $x = b$ and we specify the boundary conditions of f there, then only certain wavelengths are possible for the steady solutions, and (48) shows that these solutions make (44) stationary for variations satisfying the boundary conditions. If V is a local minimum then by (45) the steady solutions will be stable to all small enough perturbations; if V is a local maximum then after a small perturbation the fluid can never return to its original state, and if it does subsequently reach a steady state it will be at a wavelength for which V has a smaller value than originally.

The steady solutions are given by (37), in terms of the parameter λ, which may take only certain values for given boundary conditions, and it may easily be shown, by considering the shape of g or by explicit evaluation, that (47) is a decreasing function of λ, and hence wavelength. Thus perturbations cannot result in a shift to a shorter wavelength. To discover whether a larger wavelength could result, we would need to consider the second variation of (44) in terms of λ; the result might be that for certain range of λ the steady solutions are stable. Unfortunately I have not yet had time to do the calculations. This line of attack seems very promising.

(14) <u>Effect of Rotation</u>

Suppose that the Benard layer is rotating with angular velocity $\Omega \underline{z}$. We work in the rotating frame of reference and consider motions independent of y, and with a y-component of velocity $V(x,z)$. The governing equations are then

$$\left.\begin{aligned}
&\frac{\partial}{\partial t}(\nabla^2\psi) + \frac{\partial(\psi,\nabla^2\psi)}{\partial(x,z)} = \nu\nabla^4\psi - 2\Omega V_z + g\alpha\theta_x \\
&\frac{\partial V}{\partial t} + \frac{\partial(\psi,V)}{\partial(x,z)} = \nu\nabla^2 V + 2\Omega\psi_z \\
&\frac{\partial\theta}{\partial t} + \frac{\partial(\psi,\theta)}{\partial(x,z)} = \kappa\nabla^2\theta + \beta\psi_x \\
&\psi = \psi_{zz} = V_z = \theta_z = 0 \quad \text{or} \quad z = \pm d.
\end{aligned}\right\} \tag{49}$$

The dimensionless form of these is

$$\left.\begin{aligned}
&\frac{1}{\sigma}\left\{\frac{\partial}{\partial t}(\nabla^2\psi) + \frac{\partial(\psi,\nabla^2\psi)}{\partial(x,z)}\right\} = \nabla^4\psi - \lambda V_z + R\theta_x, \\
&\frac{1}{\sigma}\left\{\frac{\partial V}{\partial t} + \frac{\partial(\psi,V)}{\partial(x,z)}\right\} = \nabla^2_V + \lambda\psi_z, \\
&\frac{\partial\theta}{\partial t} + \frac{\partial(\psi,\theta)}{\partial(x,z)} = \nabla^2\theta + \psi_x, \\
&\psi = D^2\psi = Dv = D\theta = 0 \quad \text{at } z = \pm 1,
\end{aligned}\right\} \tag{50}$$

where V is measured in units of κ/d, and $\lambda = 2\Omega d^2/\nu$.

We now consider the motion when $R = R_0 + \epsilon^2$. In order for the solution to be near to that obtained without rotation, it is necessary that λ is order ϵ, and this implies that V must be order ϵ^2. Making the transformations $\lambda \longrightarrow \epsilon\lambda$, $V \longrightarrow \epsilon^2 V$, together with those given in section (5), we can proceed as before. The calculation is straightforward, and the end result is Eq.(28), except that U is defined by the equations

$$\frac{d^4u}{dz^4} = -1 - \lambda P(z), \qquad u = \frac{d^2u}{dz^2} = 0 \quad \text{or} \quad z = \pm 1 \tag{51}$$

Hence

$$\frac{\partial f}{\partial t} = a(f_x^3)_x - b f_{xxxx} - (c_1 - c_2\lambda^2) f_2 \tag{52}$$

where

$a = 1.230,.59$

$b = 0.787,157$

$c_1 = 2/15$

$c_2 = 0.164021$

This part of my report is unfortunately rather compressed. For completeness I ought to give V and show, what is in fact true, that the initially unknown function in it does not appear in the equation for f.)

If we ignore the nonlinear term in (52) and try a solution of the form

$$f(x,t) = g(t)\sin kx$$

then we require

$$\frac{dg}{dt} = k^2\left\{(c_1 - c_2\lambda^2) - bk^2\right\} g.$$

Thus a disturbance of wave number k will grow if

$$bk^2 < c_1\epsilon^2 - c_2\lambda^2 = c_1(R - R_0) - c_2 T,$$

where T is the Taylor number. Thus for given R, slightly above R_0, we can define a critical Taylor number by

$$T_c = \frac{c_1}{c_2}(R - R_0)$$

If $T > T_c$ then the system is stable. If $T < T_c$ then apart from a linear change of scale, Eq.(52) is the same as that studied in the nonrotating case, and most of the previous analysis carries over. These results are in accord with the general result that rotation is a stabilising influence.

(15) Extensions of the Present Work

It would be of great interest to devise a variational principle for this problem, expressing R as a quotient of two integrals, in the manner of Chandrasekhar (1961). This might be quite easy. A more detailed theory of the rotating Bénard layer could be given, and the theory could be extended to allow for the effect of a magnetic field, when the fluid is electrically conducting. The differential equation for f is of interest in itself, irrespective of its application to this particular problem; regarded as an initial value problem it might be possible to obtain some very general results about how the evolution of f depends on the initial value.

Acknowledgments

I should like to thank Michael Proctor for starting me off on this problem, Stephen Childress for keeping me going, and Edward Spiegel for teaching me the art of conversation; all three made incisive suggestions. In particular, Stepher Childress suggested that I consider the functional defined in section (13). I also thank Willem Malkus for running the show (his labors are appreciated by all of us) and for offering me a GFD fellowship in the first place.

References

Bowman, F. 1961 Introduction to Elliptic Functions, Dover Publications, Inc.

Chandrasekhar, S. 1961 Hydrodynamic and Hydromagnetic Stability, Oxford Univ. press.

Hurle, D.T.G., E. Jakeman and E.R.Pike 1967 On the Solution of the Bénard problem with boundaries of finite conductivity. Proc.Roy.Soc.A 296:469-475.

Malkus, W.V.R. and G. Veronis 1956 Finite amplitude cellular convection, J.F.M., 4: 225-260.

SIDEWALL BOUNDARY LAYERS IN RAPIDLY ROTATING HYDROMAGNETIC CONVECTION

Francis J. Condi

1. Introduction

Convection in the earth's outer core is generally felt to be a most likely source of energy for driving the geodynamo. An assumption is usually made that the Lorentz force is comparable with the Coriolis force, but Busse (1973) advocates that rotation is dominant in the core and that the above balance is unlikely. Busse (1970) has found that thermal instability in an internally heated, rotating, self-gravitating sphere sets in as long thin convection cells in a cylindrical annular region intersecting the sphere at about 60^{o} latitude. In his (1975) model he considered a cylindrical annulus with sloping top and bottom boundaries and found that the Ekman layers which form on these boundaries have profound influence on dynamo action.

Eltayeb (1972) concerned himself with various cases of hydromagnetic convection when both the Taylor number T, and Hartmann number M are large. These numbers are defined as

$$T = \frac{4|\Omega|^2 d^4}{\nu^2} \quad \text{and} \quad M = \frac{|B_o| d}{(\mu \rho \nu \eta)^{1/2}}$$

where Ω is angular velocity, d is a characteristic length; ν kinematic viscosity, ρ density, μ permeability, η magnetic diffusivity, and $|B_o|$ magnetic field strength. The cases are classified by the relative orientations and magnitude of rotation and magnetic field for various types of boundaries. He found when $T << M^4$ the most unstable mode has a roll axis making a small angle with the magnetic field and when $T \sim O(M^4)$, for a certain critical value, the roll axis becomes parallel to the rotation axis. In addition, for $T > M^4$ the critical Rayleigh number of the oblique rolls is greater than that of the rolls with axes parallel to rotation, hence the latter is preferred. The orientation of the cells for $T \sim O(M^4)$ and $T > M^4$ is the same as that for Busse (1970).

Another interesting result of Eltayeb's study is that in some cases the boundary conditions to be applied on the mainstream depend upon the insulating

rather kinematic properties of the boundary. The following analysis will be concerned with this question and its effect on the stability problem. A balance between the Coriolis and Lorentz force will be assumed. Therefore a several-hundred gauss toroidal field will be supposed to exist in the earth's core. The cases considered here will be for large T and large M where $T \sim O(M^4)$.

2. The Problem

Initially consider a rotating, thin annulus with a toroidal magnetic field, in which the ratio of gap width to mean radius is small enough so that local cartesian coordinates may be used. Ignoring the centrifugal force, we may treat the problem as a layer of electrically conducting fluid in solid body rotation, with a uniform magnetic field. We suppose that the temperature of the lower boundary (z = 0) is greater than that of the upper boundary (z = d) and take the case when gravity acts in the negative z-direction, rotation in the x-direction and the uniform magnetic field in the y-direction where Oxyz is a right-handed coordinate system (see Fig.1).

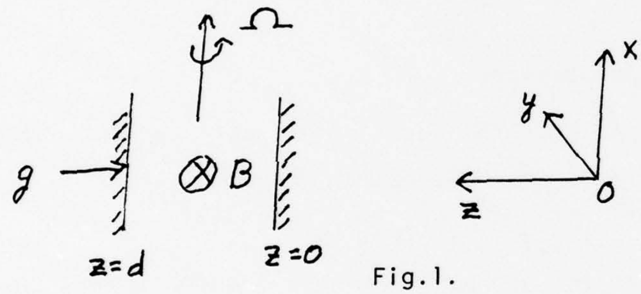


Fig.1.

The fluid has the following physical properties: electrical conductivity σ, kinematic viscosity ν, thermal diffusivity κ, permeability μ, and angular velocity Ω. An assumption is made that when the temperature gradient is large, the conduction solution will be unstable to convective motions. We suppose that squares and products of perturbation quantities, θ the perturbation temperature, b the perturbation in the magnetic field, and u the perturbation velocity are negligible. Therefore the equations are linearized around the conduction solution.

Dimensionless variables are introduced by the following transformations (Roberts and Stewartson (1974))

$$\left.\begin{aligned} x^* &= \frac{d}{\pi}x \qquad t^* = \frac{d^2}{\kappa\pi^2 t} \\ u^* &= \frac{\pi\kappa}{d}\varepsilon u(x,t) \\ B^* &= B_0\left[\hat{y} + \mu\sigma\kappa\varepsilon B(x,t)\right] \\ \theta^* &= \theta_0 + \frac{\beta d}{\pi}\left[z + \varepsilon\theta(x,t)\right] \end{aligned}\right\} \qquad (1)$$

where ε is a small parameter representing the magnitude of the disturbance applied at t = 0, β is the temperature gradient, u is the velocity, B_0 the uniform magnetic field, and Θ_0 a constant temperature. The equation of state is given by

$$\rho^* = \rho_0 [1 - \tilde{\alpha}(\theta - \theta_0)] \tag{2}$$

where $\tilde{\alpha}$ is the coefficient of volume expansion.

Upon employing the Boussinesq approximation, the following linearized equations result

Momentum:
$$\delta^2 \frac{\partial \underline{u}}{\partial t} + \lambda(\hat{x} \wedge \underline{u}) = \nabla \tilde{p} + \frac{\partial \underline{B}}{\partial y} + \lambda R \theta(x,z)\hat{z} + E\lambda\nabla^2 \underline{u} \tag{3}$$

Temperature:
$$(\nabla^2 - q\frac{\partial}{\partial t})\theta = W \tag{4}$$

Induction:
$$(\nabla^2 - q\frac{\partial}{\partial t})\underline{B} = -\frac{\partial \underline{u}}{\partial y} \tag{5}$$

where
$$\left.\begin{array}{l} \lambda = \dfrac{2\Omega\rho_0}{\sigma B_0^2}, \quad R = \dfrac{g\tilde{\alpha}\beta d^2}{2\Omega\kappa\pi^2}, \quad q = \mu\sigma\kappa \\ \delta^2 = \dfrac{\lambda\pi^2\kappa}{2\Omega d^2}, \quad E = \dfrac{\nu\pi^2}{2\Omega d^2} \end{array}\right\} \tag{6}$$

and $\tilde{p}$ is a reduced pressure.

We now take the z-components of the curl of (3), curl curl of (3) and curl of (5). It is then possible to assume a normal modes solution

$$G(x,y,z,t) = G(z)e^{i(kx + ly + \sigma t)} \tag{7}$$

where G represents any of the above variables. The result is five ordinary differential equations. It may be shown that these equations may be written as

$$(D^2 - a^2 - i\sigma)\theta = W \tag{8}$$

$$(D^2 - a^2 - iq\sigma)B = -ilW \tag{9}$$

$$\{[(i\delta^2\sigma - E\lambda(D^2 - a^2))(D^2 - a^2 - iq\sigma)] - l^2\}\zeta = i\lambda k(D^2 - a^2 - iq\sigma)W \tag{10}$$

$$\{[(i\delta^2\sigma - E\lambda(D^2 - a^2))(D^2 - a^2 - iq\sigma)] - l^2]\xi = l\lambda k W \tag{11}$$

$$\begin{aligned} &\{(D^2-a^2)[(D^2-a^2) - ip^{-1}\sigma](D^2-a^2-iq\sigma)(D^2-a^2-i\sigma) + l^2M^2(D^2-a^2)(D^2-a^2-i\sigma)\} \\ &\times\{[(ip^{-1}\sigma - (D^2-a^2))(D^2-a^2-iq\sigma)] - M^2l^2\}W + \lambda k^2 M T^{1/2}(D^2-a^2-iq\sigma)^2(D^2-a^2-i\sigma)W \\ &= \lambda M^2 R a^2 (D^2-a^2-iq\sigma)\{[(ip^{-1}\sigma - (D^2-a^2))(D^2-a^2-iq\sigma)] - M^2l^2\}W \end{aligned} \tag{12}$$

where $D = \frac{d}{dz}$, $a^2 = k^2 + l^2$, $p = \nu/\kappa$

and ζ , ξ and B are the vertical components of vorticity, electric current, and magnetic field respectively. We note that $E\lambda$ may be expressed as M^{-2} and E may be expressed as $T^{-1/2}$.

Upon assuming the principle of exchange of stabilities is valid ($\sigma = 0$) and canceling a $(D^2 - a^2)$ from each side of (12) Eq.(8) through (12) reduce to

$$(D^2-a^2)\Theta = W \tag{13}$$

$$(D^2-a^2)B = -i\ell W \tag{14}$$

$$[E\lambda(D^2-a^2)^2+\ell^2]\zeta = -ik\lambda(D^2-a^2)W \tag{15}$$

$$[E\lambda(D^2-a^2)+\ell^2]\xi = -\ell k\lambda W \tag{16}$$

$$\{(D^2-a^2)[E\lambda(D^2-a^2)^2+\ell^2]^2 - k^2\lambda^2(D^2-a^2)^2\}W = Ra^2\lambda[E\lambda(D^2-a^2)^2+\ell^2]W \tag{17}$$

3. Boundary Conditions

Equation (17) is a tenth order equation. We will need five boundary conditions on each of the two boundaries for its solution. We follow Eltayeb (1972) and assume for all kinds of boundaries that they are perfect thermal conductors, therefore the temperature perturbation vanishes there. We also assume that the normal velocity vanishes at the boundary. Then at $z = 0, d$ $\theta = W = 0$. The next conditions applied are that for rigid boundaries (no slip) $DW = 0$ and for free boundaries $D^2W = 0$. The magnetic boundary conditions are

$$\langle n \cdot B \rangle = 0, \quad \langle n_\wedge E \rangle, \quad \langle n \cdot J \rangle = 0 \tag{18}$$

where $\langle\ \rangle$ denotes a jump in the quantity across the boundary. It can be shown that for an insulating boundary the condition becomes $\xi = 0$ at $z = 0, d$, and for a perfect electrical conductor that $D\xi = 0$ at $z = 0, d$. The boundary conditions may be summarized as follows:

(1) Free, Perfectly Conducting Boundaries

$$\theta = W = D^2\zeta = D\xi = 0 \quad at \quad z = 0, d \tag{19}$$

(ii) Rigid, Perfectly Conducting Boundaries

$$\theta = W = DW = \zeta = D\xi = 0 \quad at \ z = 0, d \tag{20}$$

(iii) Free, Insulating Boundaries

$$\theta = W = D^2W = D\zeta = \xi = 0 \quad at \ z = 0, d \tag{21}$$

(iv) Rigid, Insulating Boundaries

$$\theta = W = DW = \zeta = \xi = 0 \quad at \ z = 0, d \tag{22}$$

For all cases B is continuous.

4. The Boundary Layer

From (17) we obtain a boundary layer equation

$$D^4(D^6 - k^2 T)W = 0 \tag{23}$$

and a mainstream equation

$$\{\ell^4(D^2 - a^2) - k^2\lambda^2(D^2 - a^2)^2 - Ra^2\lambda\ell^2\}W = 0 \tag{24}$$

It should be noted that no magnetic terms appear in the boundary layer equation (23). The thickness of the layer is $O(T^{-1/6})$.

A solution to (23) is

$$W = A_1 e^{-\alpha z} + A_2 e^{q_1\alpha z} + A_3 e^{q_2\alpha z} + B_0 + B_1 z + B_2 z^2 + B_3 z^3 \tag{25}$$

where

$$q_1 = -\tfrac{1}{2} + \tfrac{\sqrt{3}}{2}i, \quad q_2 = -\tfrac{1}{2} - \tfrac{\sqrt{3}}{2}i \tag{26}$$

$$\alpha = T^{1/6} k^{1/3}$$

Expressions for θ, ζ, ξ can now be found from (13), (15) and (16) respectively. We now wish to test the hypothesis that $\theta = W = 0$ are the correct boundary conditions to apply at the edge of the mainstream. At $z = 0$, for any boundary

$$\theta(0) = \alpha^{-2}A_1 + \alpha^{-2}q_1 A_2 + \alpha^{-2}q_2 A_3 + \theta_{MS} = 0 \tag{27}$$

$$W(0) = A_1 + A_2 + A_3 + W_{MS} = 0 \tag{28}$$

where the subscript MS denotes mainstream quantities.

The first case to consider is that of the free, perfectly conducting boundary. The boundary conditions are given by (19). The other equations which are required are

$$D^2W(0) = \alpha^2 A_1 + \alpha^2 q_1^2 A_2 + \alpha^2 q_2^2 A_3 + (D^2W)_{MS} = 0 \tag{29}$$

$$D\zeta(0) = -\alpha^2 A_1 + \alpha^2 q_1^2 A_2 + \alpha^2 q_2^2 A_3 + (D\zeta)_{MS} = 0 \tag{30}$$

$$D\xi(0) = A_1 - A_2 - A_3 + (D\xi)_{MS} = 0 \tag{31}$$

(27) through (31) may be rewritten as

$$A_1 + q_1 A_2 + q_2 A_3 = -\alpha^2\theta_{MS} = -\alpha^2 V \tag{32}$$

$$A_1 + A_2 + A_3 = -W_{MS} = U \tag{33}$$

$$A_1 + q_1^2 A_2 + q_2^2 A_3 = -(D^2W)_{MS}/\alpha^2 = 0 = X/\alpha^2 \tag{34}$$

$$-A_1 + q_1^2 A_2 + q_2^2 A_3 = -(D\zeta)_{MS}/\alpha^2 = 0 = Y/\alpha^2 \tag{35}$$

$$A_1 - A_2 - A_3 = -(D\xi)_{MS} = Z \tag{36}$$

where U, V, X, Y, Z are defined for convenience by the above equations. In addition, we may form a relation between (33) and (36)

$$(Z-U)A_1 + (Z+U)A_2 + (Z+U)A_3 = 0 \tag{37}$$

θ_{ms} is just the left-hand side of (32) divided by a large quantity. Therefore to order (α^{-1}) it is zero and we may concentrate on the other equations. No similar statement can be made about the other quantities. From (33), (34) and (35) the coefficients A_1, A_2 and A_3 may be determined

$$A_1 = 0,\ A_2 = \frac{-Uq_3^2}{q_1^2 - q_2^2},\quad A_3 = \frac{Uq_1^2}{q_1^2 - q_2^2} \tag{38}$$

Using (37) a relation may be obtained between Z and U

$$U(Z+U) = 0 \tag{39}$$

U = 0 is just the trivial solution. All three coefficients in this case are zero. The interesting relation which emerges is that Z + U = 0 or

$$W_{ms} + (D\xi)_{ms} = 0 \tag{40}$$

This says that the vertical velocity alone does not vanish at the edge of the mainstream but that a linear combination of the vertical velocity and change in vertical current are brought to zero there. $(D\xi)_{ms}$ can be expressed in terms of W_{ms} by use of (16).

We now consider the case of rigid, perfectly conducting boundaries. The boundary conditions are given by (20). Equations (32), (33), (36) and (37) are still valid. Instead of (40) and (41) the following equations are used

$$-A_1 + q_1 A_2 + q_2 A_3 = -(DW)_{ms}/\alpha = 0 = X/\alpha \tag{41}$$

$$A_1 + q_1 A_2 + q_2 A_3 = -\zeta_{ms}/\alpha = 0 = Y/\alpha \tag{42}$$

By following the same analysis as that for the previous case, the result is

$$W_{ms} + (D\xi)_{ms} = 0 \tag{43}$$

The next case is that of free, insulating boundaries. The boundary conditions to be applied are (21). Equations (32) through (35) are still valid. Equation (36) is replaced by

$$-A_1 + q_1^2 A_2 + q_2^2 A_3 = -\alpha\,\xi_{ms} = \alpha Z. \tag{44}$$

From this analysis it can be seen that Z is zero to order (1), i.e.

$$\xi_{ms} = 0 \tag{45}$$

The same result holds for rigid insulating boundaries.

The results for all cases indicate that the boundary conditions to be

applied on the mainstream do indeed depend on the conductivity properties of the boundary; however, the above derived boundary conditions differ from those of Eltayed (1972) who assumed $W = \Theta = 0$ to be valid on the mainstream. From this analysis we can expect a significant change in the critical Rayleigh numbers. Extension of the work sill be to calculate the critical Rayleigh numbers with the derived boundary conditions and consider more physically realizable geometries.

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References

1. Busse, F.H. 1970 Instabilities in Rapidly Rotating Systems. J.Fluid Mech. 44: 441-460.
2. Busse, F.H. 1973 Generation of Magnetic Fields by Convection. J.Fluid Mech. 57: 529-544.
3. Busse, F.H. 1975 A Model of the Geodynamo. Geophys.J.Roy.Astr.Soc. 42" 437-459.
4. Eltayeb, I.A. 1972 Hydromagnetic Convection in a Rapidly Rotation Fluid Layer. Proc.Roy.Soc. A 326: 229-254.
5. Roberts, P.H. and K.Stewartson 1974 On Finite Amplitude Convection in a Rotating Magnetic System. Phil.Trans.Roy.Soc. A 277: 287-315.

LONG WAVE MOTIONS IN AN ADIABATIC ATMOSPHERE

Pham Giem Cuong

Introduction

We use the tidal equations, derived by Karal* with some minor modifications of our own, the small amplitude wave-like disturbances in a spherical adiabatic atmosphere. It is found that to the first order approximation there is only one band symmetric with respect to the equation where disturbances of a given frequency ω can propagate. For $\omega \lesssim \Omega$, where Ω is the rotation of the planet, the band width is proportional to Ω. For larger ω it is proportional to ω. Inside this region only waves with azimuthal wavelengths satisfying $m^2 \lesssim \frac{\gamma}{\gamma-1}\omega^2$ can propagate where γ is the adiabatic exponent; this smaller $|m|$ is the farthest the wave can reach poleward. The characteristics of the waves are examined qualitatively.

*The tidal equations used in this paper are only one part of Karal's work on the weather. Since we do not know when it will be published, an outline of his derivation is given here.

1. Karal's Model

We consider a non-viscous, non-heat conducting isentropic troposphere surrounding a rigid, rotating and almost spherical planet. The boundary between the troposphere and the tropopause is considered to be a free surface where the pressure $\mathcal{P}_T$ is given. This boundary and the surface of the solid planet are described respectively by

$$r = \eta(\theta, \varphi, t) \tag{1.1}$$

and

$$r = h(\theta, \varphi) \tag{1.2}$$

where (r, θ, φ) are the spherical coordinates with origin at the center of the planet and t is the time. Let us call the mean radius of the solid planet and the mean thickness of the troposphere d. Then

$$\epsilon = \frac{d}{a} \tag{1.3}$$

is a very small number which is of the order 10^{-3} in the earth's case. This is the case in which we are most interested. For later use we also defined

$$\xi(\theta, \varphi, t) = \eta(\theta, \varphi, t) - a \tag{1.4}$$

$$\sigma(\theta, \varphi) = h(\theta, \varphi) - a \tag{1.5}$$

and

$$\chi = r - a. \tag{1.6}$$

The equations describing such an atmosphere are

$$\frac{d\underset{\sim}{u}}{dt} = -2\Omega \times \underset{\sim}{u} - \underset{\sim}{\Omega} \times (\underset{\sim}{\Omega} \times \underset{\sim}{r}) + \underset{\sim}{G} - \frac{1}{\rho}\nabla p \tag{1.7}$$

$$\frac{d\rho}{dt} + \rho \nabla \cdot \underset{\sim}{u} = 0 \tag{1.8}$$

$$\frac{d}{dt}\frac{p}{\rho^{\gamma}} = 0 \tag{1.9}$$

The appropriate boundary conditions are

$$\underset{\sim}{u} \cdot \underset{\sim}{n}_{\eta} = \frac{\partial \eta}{\partial t} \quad \text{at} \quad r = \eta \tag{1.10}$$

$$\underset{\sim}{u} \cdot \underset{\sim}{n}_{h} = 0 \quad \text{at} \quad r = h \tag{1.11}$$

and

$$p = \mathcal{P}_T \quad \text{at} \quad r = \eta \tag{1.12}$$

Here $\underset{\sim}{u}$, $\underset{\sim}{G}$, p, ρ, $\underset{\sim}{n}_{\eta}$, $\underset{\sim}{n}_{h}$ are respectively the velocity, gravity, pressure, density, and unit normal vectors to η and h, see Fig.1 The equations (7, 8 and 9) are the momentum, continuity and isentropy equations. The Eq.(10) describes the free surface $r = \eta$ while Eq.(11) corresponds to the rigid one $r = h$.

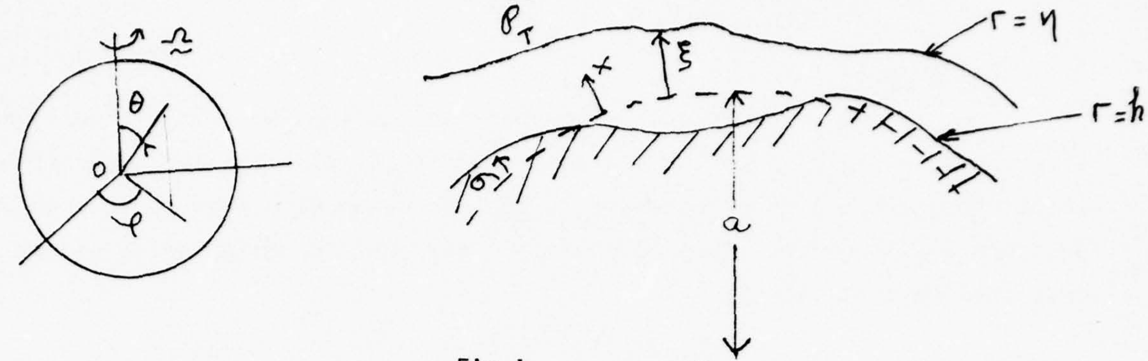

Fig.1

Let

$$\underset{\sim}{u} = (u, v, w)$$

and

$$\underset{\sim}{G} = (G_r, G_\theta, G_\varphi)$$

in spherical coordinates. Then the Eqs.(7-12) become

$$\frac{du}{dt} - \frac{v^2+u^2}{r} = G_r + \Omega^2 r \sin^2\theta + 2\Omega w \sin\theta - \frac{1}{\rho}\frac{\partial p}{\partial r} \tag{1.13}$$

$$\frac{dv}{dt} + \frac{uv}{r} - \frac{w^2\cot\theta}{r} = G_\theta + \Omega^2 r \sin\theta\cos\theta + 2\Omega w\cos\theta - \frac{1}{r\rho}\frac{\partial p}{\partial\theta} \tag{1.14}$$

$$\frac{dw}{dt} + \frac{uw}{r} + \frac{vw\cot\theta}{r} = G_\varphi - 2\Omega u\sin\theta - 2\Omega v\cos\theta - \frac{1}{r\rho\sin\theta}\frac{\partial p}{\partial\varphi} \tag{1.15}$$

$$\frac{d\rho}{dt} - \rho\left[\frac{\partial u}{\partial r} + \frac{1}{r}\frac{\partial v}{\partial\theta} + \frac{1}{r\sin\theta}\frac{\partial w}{\partial\varphi} + \frac{2u}{r} + \frac{v\cot\theta}{r}\right] = 0 \tag{1.16}$$

$$\rho\frac{dp}{dt} = \gamma p\frac{d\rho}{dt} \tag{1.17}$$

$$u - \frac{v}{r}\frac{\partial\eta}{\partial\theta} - \frac{w}{r\sin\theta}\frac{\partial\eta}{\partial\varphi} - \frac{\partial\eta}{\partial t} = 0 \quad \text{at } r = \eta \tag{1.18}$$

$$u - \frac{v}{r}\frac{\partial h}{\partial\theta} - \frac{w}{r\sin\theta}\frac{\partial h}{\partial\varphi} = 0 \quad \text{at } r = h \tag{1.19}$$

$$p = p_T \qquad \text{at } r = \eta \tag{1.20}$$

The Eqs.(13-18) with the boundary conditions (19,20) form a system for the 6 unknowns u, v, w, p, ρ, η for given G_r, G_θ, G_φ and intral values of the unknowns.

We now nondimensionalize the equations by defining

$$\begin{aligned}
&\bar{x} = \frac{x}{d}, \quad \bar{r} = 1 + \epsilon\bar{x}, \quad \bar{t} = \frac{\sqrt{gd}}{a}t \\
&\bar{u} = \frac{u}{\epsilon\sqrt{gd}}; \quad \bar{v} = \frac{v}{\sqrt{gd}}; \quad \bar{w} = \frac{w}{\sqrt{gd}} \\
&\bar{p} = \frac{p}{\rho_0 gd}, \quad \bar{\rho} = \frac{\rho}{\rho_0} \\
&\bar{\sigma} = \frac{\sigma}{d}, \quad \bar{\xi} = \frac{\xi}{d}, \quad \bar{\Omega} = \frac{2a\Omega}{\sqrt{gd}} \\
&\bar{G}_r = \frac{G_r}{g}, \quad \bar{G}_\theta = \frac{G_\theta}{\epsilon g}, \quad \bar{G}_\varphi = \frac{G_\varphi}{\epsilon g}
\end{aligned} \tag{1.21}$$

where g is the vertical acceleration of gravity at $r = a$ and ρ_0 A mean

density. This scaling procedure makes the quantities in the vertical direction comparable to those in the horizontal. The horizontal velocity has been scaled by the shallow water wave velocity, since we will be interested only in motions of wavelengths long compared to d (but still small compared to a).

The next step is to expand all the dimensionless (dropped bars) variables u, v, w, p, ρ and ξ in series of the form, e.g.,

$$u = \sum_{n=0}^{\infty} \epsilon^n u^{(n)}, \tag{1.22}$$

to the zeroth order in ϵ we get from (13-20)

$$\frac{1}{\rho^{(0)}} \frac{\partial p^{(0)}}{\partial x} - G_x^{(0)} = 0 \tag{1.23}$$

$$\left[\frac{\partial}{\partial t} + u^{(0)}\frac{\partial}{\partial x} + v^{(0)}\frac{\partial}{\partial \theta} + \frac{w^{(0)}}{\sin\theta}\frac{\partial}{\partial \varphi}\right] v^{(0)} - w^{(0)2}\cot\theta - \frac{\Omega^2}{4}\sin\theta\cos\theta - \Omega w^{(0)}\cos\theta + \frac{1}{\rho^{(0)}}\frac{\partial p^{(0)}}{\partial\theta} - G_\theta^{(0)} = 0 \tag{1.24}$$

$$\left[\frac{\partial}{\partial t} + u^{(0)}\frac{\partial}{\partial x} + v^{(0)}\frac{\partial}{\partial \theta} + \frac{w^{(0)}}{\sin\theta}\frac{\partial}{\partial \varphi}\right] w^{(0)} + v^{(0)}w^{(0)}\cot\theta - \Omega v^{(0)}\cos\theta + \frac{1}{\rho^{(0)}\sin\theta}\frac{\partial p^{(0)}}{\partial \varphi} - G_\varphi^{(0)} = 0 \tag{1.25}$$

$$\frac{\partial}{\partial x}\left(u^{(0)}\rho^{(0)}\right) + \frac{\partial \rho^{(0)}}{\partial t} + v^{(0)}\frac{\partial \rho^{(0)}}{\partial \theta} + \frac{w^{(0)}}{\sin\theta}\frac{\partial \rho^{(0)}}{\partial \varphi} + \rho^{(0)}\frac{\partial v^{(0)}}{\partial \theta} + \frac{\rho^{(0)}}{\sin\theta}\frac{\partial w^{(0)}}{\partial \varphi} + \rho^{(0)}v^{(0)}\cot\theta = 0 \tag{1.26}$$

$$\left[\frac{\partial}{\partial t} + u^{(0)}\frac{\partial}{\partial x} + v^{(0)}\frac{\partial}{\partial \theta} + \frac{w^{(0)}}{\sin\theta}\frac{\partial}{\partial \varphi}\right] \mathrm{Ln}\, \frac{p^{(0)}}{\rho^{(0)\gamma}} = 0 \tag{1.27}$$

$$u^{(0)} - v^{(0)}\frac{\partial \xi^{(0)}}{\partial \theta} - \frac{w^{(0)}}{\sin\theta}\frac{\partial \xi^{(0)}}{\partial \varphi} - \frac{\partial \xi^{(0)}}{\partial t} = 0 \quad \text{at} \quad x = \xi^{(0)} \tag{1.28}$$

$$u^{(0)} - v^{(0)}\frac{\partial \sigma}{\partial \theta} - \frac{w^{(0)}}{\sin\theta}\frac{\partial \sigma}{\partial \varphi} = 0 \quad \text{at} \quad x = \sigma \tag{1.29}$$

$$p^{(0)} = P_T \qquad \text{at} \quad x = \xi^{(0)} \tag{1.30}$$

Equation (23) implies a vertical hydrostatic balance.
Equation (27) is satisfied if

$$\frac{p^{(0)}}{\rho^{(0)\gamma}} = K^{-\gamma} = \text{const.} \tag{1.31}$$

This means that the troposphere is adiabatic (or, in other words, constant entropy).

Since $G_x^{(0)} = -1$, the equations (23, 30 and 31) give

$$p^{(0)} = \left[\alpha\left(\xi^{(0)} - x\right) + \beta\right]^{\frac{\gamma}{\gamma-1}} \tag{1.32}$$

and

$$\rho^{(0)} = K\left[\alpha\left(\xi^{(0)} - x\right) + \beta\right]^{\frac{1}{\gamma-1}} \tag{1.33}$$

in which only $\xi^{(0)}$ remains unknown and

$$K = \left(\rho_0 g d / p_0\right)^{\frac{1}{\gamma}}; \quad \alpha = \frac{K}{\gamma^*}; \quad \gamma^* = \gamma/(\gamma-1); \quad \beta = P_T^{\frac{1}{\gamma^*}}. \tag{1.34}$$

Integrating (26) gives

$$u^{(0)}\rho^{(0)} = -\int_0^x \left\{\frac{\partial\rho^{(0)}}{\partial t} + v^{(0)}\frac{\partial\rho^{(0)}}{\partial\theta} + \frac{w^{(0)}}{\sin\theta}\frac{\partial\rho^{(0)}}{\partial\varphi}\right\}dx' - \int_0^x \rho^{(0)}\left\{\frac{\partial v^{(0)}}{\partial\theta} + \frac{1}{\sin\theta}\frac{\partial w^{(0)}}{\partial\varphi} + v^{(0)}\cot\theta\right\}dx' + \left[u^{(0)}\rho^{(0)}\right]_{x=0} \quad (1.35)$$

Now if we assume a spherical solid planet then $\sigma \equiv 0$ and from (29) $u^{(0)} = 0$ as $x = 0$. Consequently the last term on the right-hand side of (35) vanishes. Eliminating $\rho^{(0)}$ in (35) by using (33) gives

$$u^{(0)}\left[\alpha(\xi^{(0)}-x)+\beta\right]^{\frac{1}{\gamma-1}} = -\int_0^x \left[\alpha(\xi^{(0)}-x')+\beta\right]^{\frac{2-\gamma}{\gamma-1}}\cdot\left[\frac{\alpha}{\gamma-1}L_1 + \frac{1}{\gamma}L_1^*\right]dx' - \int_0^x \left[\alpha(\xi^{(0)}-x')+\beta\right]^{\frac{1}{\gamma-1}}\cdot L_2\,dx' \quad (1.36)$$

where

$$\left.\begin{aligned}
L_1 &= \left[\frac{\partial}{\partial t} + v^{(0)}\frac{\partial}{\partial\theta} + \frac{w^{(0)}}{\sin\theta}\frac{\partial}{\partial\varphi}\right]\xi^{(0)} \\
L_1^* &= P_T^{-\frac{1}{\gamma}}\cdot\left[\frac{\partial}{\partial t} + v^{(0)}\frac{\partial}{\partial\theta} + \frac{w^{(0)}}{\sin\theta}\frac{\partial}{\partial\varphi}\right]P_T \\
L_2 &= \frac{\partial v^{(0)}}{\partial\theta} + \frac{1}{\sin\theta}\frac{\partial w^{(0)}}{\partial\varphi} + v^{(0)}\cot\theta
\end{aligned}\right\} \quad (1.37)$$

Let $x = \xi^{(0)}$ in (36) and solve for $u^{(0)}$. Substituting this result into (28) we get

$$\int_0^{\xi^{(0)}}\left[\alpha(\xi^{(0)}-x')+\beta\right]^{\frac{2-\gamma}{\gamma-1}}\cdot\left[\frac{\alpha}{\gamma-1}L_1 + \frac{1}{\gamma}L_1^*\right]dx' + \int_0^{\xi^{(0)}}\left[\alpha(\xi^{(0)}-x')+\beta\right]^{\frac{1}{\gamma-1}}\cdot L_2\,dx' + \beta^{\frac{1}{\gamma-1}}L_1 = 0$$

Also if we assume now that $v^{(0)}$ amd $w^{(0)}$ are independent of height x then this becomes

$$kL_1 + \left[\alpha\xi^{(0)}+\beta\right]L_2 - \left[\alpha\xi^{(0)}+\beta\right]^{-\frac{1}{\gamma-1}}\cdot\left[P_T L_2 + P_T^{\frac{1}{\gamma}}L_1^*\right] + P_T^{\frac{1}{\gamma}}L_1^* = 0$$

This becomes simpler if we take $P_T = 0$, namely

$$\gamma^* L_1 - \xi^{(0)}L_2 = 0. \quad (1.38)$$

In brief we have assumed an inviscid, non-heat conducting, compressible, adiabatic atmosphere surrounding a rotating spherical rigid planet. The top of the atmosphere is a free surface where the pressure is zero, and the horizontal velocity in the atmosphere is height-independent. The system of equations describing such an atmosphere is

$$\left[\frac{\partial}{\partial t} + v\frac{\partial}{\partial\theta} + \frac{w}{\sin\theta}\frac{\partial}{\partial\varphi}\right]v - w^2\cot\theta - \frac{\Omega^2}{4}\sin\theta\cos\theta - \Omega w\cos\theta + \frac{\partial\xi}{\partial\theta} - G_\theta = 0 \quad (1.39)$$

$$\left[\frac{\partial}{\partial t} + v\frac{\partial}{\partial\theta} + \frac{w}{\sin\theta}\frac{\partial}{\partial\varphi}\right]w + vw\cot\theta - \Omega v\cos\theta + \frac{1}{\sin\theta}\frac{\partial\xi}{\partial\varphi} - G_\varphi = 0 \quad (1.40)$$

$$\gamma^*\left[\frac{\partial}{\partial t} + v\frac{\partial}{\partial\theta} + \frac{w}{\sin\theta}\frac{\partial}{\partial\varphi}\right]\xi + \xi\left[\frac{\partial v}{\partial\theta} + \frac{1}{\sin\theta}\frac{\partial w}{\partial\varphi} + v\cot\theta\right] = 0 \quad (1.41)$$

$$u = -\frac{x}{\gamma^*}\left[\frac{\partial v}{\partial\theta} + \frac{1}{\sin\theta}\frac{\partial w}{\partial\varphi} + v\cot\theta\right] \quad (1.42)$$

$$p = \left[\frac{K}{\gamma^*}\left(\xi - x\right)\right]^{\gamma^*} \tag{1.43}$$

$$\rho = K\left[\frac{K}{\gamma^*}\left(\xi - x\right)\right]^{\frac{1}{\gamma-1}} \tag{1.44}$$

Here the superscript (o) has been dropped.

This derivation is due to Karal. The minor modifications we have introduced are:

(i) the scaling of $u = \epsilon\sqrt{gd}$ instead of his $\bar{u} = \frac{\epsilon u}{\sqrt{gd}}$. This is more reasonable since u is of $O(\epsilon)$ compared to v, w, and it makes the scheme work faster without changing the results;

(ii) introduction of the operator L_1^* in (37) which, we think, he had forgotten. This does not matter in the last step since P_T is supposed to be zero anyway.

(iii) Retaining $G_\theta^{(o)}$ and $G_\varphi^{(o)}$ in (39, 40). In his version, after the assumption of a spherical solid planet, which he called perfect, i.e., spherical surface with spherically symmetric distribution of mass inside, he had to take $G_\theta^{(o)} = G_\varphi^{(o)} = 0$. This would lead to a distratious distribution of pressure in the static state which will be discussed in the following.

II. Allowed Regions for Linear Disturbances

The system (1.39 - 1.44) admits a static solution

$$\begin{aligned} &u = v = w = 0\,;\quad \xi = 1 \\ &p = \left[\frac{K}{\gamma^*}(1-x)\right]^{\gamma^*} \\ &\rho = K\left[\frac{K}{\gamma^*}(1-x)\right]^{\frac{1}{\gamma-1}} \end{aligned} \tag{II.1}$$

provided that

$$G_\theta = -\frac{\Omega^2}{4}\sin\theta\cos\theta\,;\quad G_\varphi = 0.$$

In this static state the top of the atmosphere is a spherical surface and so are the surfaces of constant pressure and density. The non-vanishing value for in this case is supposedly one to some non-spherically symmetric distribution of mass inside the solid planet.

Had we taken $G_\theta = 0$, as Karal did, we would have had from (1.39,40)

$$\frac{\partial\xi}{\partial\theta} = \frac{\Omega^2}{4}\sin\theta\cos\theta$$

$$\frac{\partial\xi}{\partial\varphi} = 0$$

This has the solution $\xi = \frac{1}{2}\left(3 - \frac{\Omega^2}{4}\cos^2\theta\right)$

Let us take the case of the earth where $\Omega \simeq \pi$ and $\gamma^* \simeq$ 3.5. Then this expression for ξ would give an oblate spheriodal-like atmosphere with the thickness at the equator about 3 times that at the poles. The ratio of the surface pressures between the equator and the poles would be then, using (1.43), about 4.6. Such a super-unrealistic ratio is partly caused by the assumption of homentropy and partly because of $G_\theta = 0$. This is the reason why we reshape the atmosphere by letting G_θ balance the centrifugal force.

We are now interested in perturbing the static state (1) by small amplitude disturbances. We only have to concentrate on the three equations (1.39-41) since they form an independent system for (v, w, ξ). Once these are known u, p and ρ follow by using (1.42-44). Let us call the disturbances by (v', w', ξ), i.e., in (1.39-41). Let

$$v = v', \quad w = w', \quad \xi = 1 + \xi'.$$

By neglecting quadratic terms in them we get the following linearized system:

$$\frac{\partial v'}{\partial t} = \Omega w' \cos\theta + \frac{\partial \xi'}{\partial \theta} = 0 \tag{11.2}$$

$$\frac{\partial w'}{\partial t} = \Omega v' \cos\theta + \frac{1}{\sin\theta}\frac{\partial \xi'}{\partial \varphi} = 0 \tag{11.3}$$

$$\gamma^* \frac{\partial \xi'}{\partial t} + \frac{\partial v'}{\partial \theta} + \frac{1}{\sin\theta}\frac{\partial w'}{\partial \varphi} + v' \cos + \theta = 0 \tag{11.4}$$

In (2, 3) we have the horizontal momentum balance between inertia, Coriolis and pressure forces. Equation (4) is the continuity equation, balancing the horizontal divergence of the flow with the change of height of the atmosphere.

We are going to look for solutions of (3-4) of the form

$$\begin{bmatrix} v' \\ w' \\ \xi' \end{bmatrix} = \begin{bmatrix} \mathcal{V}(\theta) \\ \mathcal{W}(\theta) \\ \Theta(\theta) \end{bmatrix} e^{i[\omega t + m\varphi + \int^{\theta}(K + i\alpha)d\theta']} \tag{11.5}$$

The fact that there are solutions proportional to $e^{i(\omega t + m\varphi)}$ is clear since the equations are linear and the coefficients do not depend on t and φ explicitly. Moreover ω and m must be real because the problem is nondissipative and axially symmetric. Hence w, m, k, α are all real and the amplitudes V, W, Θ depend only on θ . Also since we expect (5) to have a wave-like form we assume that the derivatives of V, W, Θ with respect to θ are much smaller than $|\omega|$, $|m|$ and $|K + i\alpha|$. This assumption will be justified at the end. Then (2) - (5) give

$$\begin{bmatrix} i\omega & -\Omega\cos\theta & i(K+i\alpha) \\ -\Omega\cos\theta & i\omega & \frac{i}{\sin\theta} \\ i(K+i\alpha)+\cot\theta & \frac{im}{\sin\theta} & i\gamma^*\omega \end{bmatrix} \begin{bmatrix} V \\ W \\ \Theta \end{bmatrix} = 0$$

To have a non-trivial solution for (V, W, Θ) the determinant of coefficients has to vanish; this gives

$$(2\Omega m+\omega)(K+i\alpha)\cot\theta - i\left[\gamma^*\omega^3 + m\Omega\cot\theta - (K+i\alpha)^2\omega + \gamma^*\omega\Omega^2\cos^2\theta - \frac{m^2\omega}{\sin^2\theta}\right] = 0$$

By separating the real and imaginary parts and solving for k and α we get, for $K \neq 0$ and $\omega \neq 0$,

$$\alpha - \left(\frac{m\Omega}{\omega} + \frac{1}{2}\right)\cot\theta = 0 \qquad (11.6)$$

and

$$G \equiv K^2 - \gamma^*(\omega^2+\Omega^2) - \left[\left(\frac{m\Omega}{\omega}\right)^2 + \frac{1}{4}\right] + \gamma^*\Omega^2\sin^2\theta + \left[m^2\left(1+\frac{\Omega^2}{\omega^2}\right) + \frac{1}{4}\right]\frac{1}{\sin^2\theta} = 0 \qquad (11.7)$$

For the time being we are not interested in (6) since it just changes the amplitudes of the waves by the multiplying function

$$f(\theta) = (\sin\theta)^{-\left(\frac{m\Omega}{\omega} + \frac{1}{2}\right)} \qquad (11.8)$$

which does not affect the behavior of the wave pattern.

The constraint that k be real implies

$$K^2 \geq 0$$

or from (7)

$$F(\omega,\theta) \equiv \frac{(4\gamma^*\omega^2+1)\sin^2\theta + \gamma^*\Omega^2\sin^2 2\theta - 1}{4\left(1+\frac{\Omega^2}{\omega^2}\cos^2\theta\right)} \geq m^2 \qquad (11.9)$$

For a given ω the variation of F as a function of θ is described by Fig.2. In Fig.2 it is clear that waves of a given frequency ω are confined to the region $C(\omega)$ defined by $\theta_0 < \theta < \theta_0 + \frac{\pi}{2}$. In other words, there exist in general two polar caps in which waves of given ω are not allowed to propagate. The approximate values of $\theta_0(\omega)$ are given as follows:

$$\theta \simeq \frac{1}{2\Omega\sqrt{2\gamma^*}} \simeq 3^\circ \quad \text{for } \omega \lesssim \Omega$$

and

$$\theta \simeq \frac{1}{2\omega\sqrt{2\gamma^*}} \quad \text{for } \omega \gg \Omega$$

The numerical value of 3^0 is for the earth's case. Thus the larger ω is the smaller the caps, which disappear as $\omega \rightarrow \infty$. This means that waves of high frequencies can propagate all the way to the poles, which is to be expected since rotation Ω can be neglected in this case.

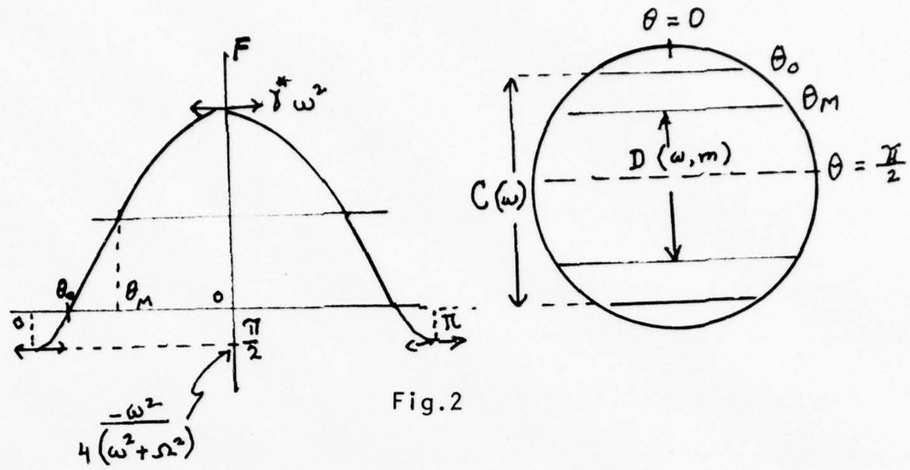

Fig.2

Now after ω has been given, suppose we also specify m, the zonal wave number. Then the waves are confined in the smaller region $D(\omega, m)$ defined by $\theta_M < \theta < \theta_M + \frac{\pi}{2}$ where $\theta_0 < \theta_M < \frac{\pi}{2}$ due to (9). This region $D(\omega, m)$ will become narrower as m^2 gets larger and will disappear when m^2 reaches its maximum possible value $\gamma^* \omega^2$.

By definition

$$K^2 = F - m^2. \tag{II.10}$$

Therefore K vanishes at $\theta = \theta_M$ or $\theta = \theta_M + \frac{\pi}{2}$ and increases monotonically to its maximum value $(\gamma^* \omega^2 - m^2)$ at $\theta = \frac{\pi}{2}$. Since we have assumed that $|K|$ be large enough so that we can neglect the derivatives of V, W, Θ, any results near these borders are doubtful. Anyway θ_M can be calculated by

$$\sin^2 \theta_M = \frac{B - \sqrt{B^2 - 4\gamma^* \Omega^2 [m^2 (1 + \frac{\Omega^2}{\omega^2}) + \frac{1}{4}]}}{2\gamma^* \Omega^2}$$

where

$$B = \gamma^* (\omega^2 + \Omega^2) + (m\Omega/\omega)^2 + \frac{1}{4}.$$

III. <u>Qualitative Properties of Responses</u>

The dispersion relation defined by (II.7) can also be written as

$$G(\theta, \varphi, t, s, K, m, \omega) = 0 \tag{III.1}$$

where

$$S(\theta, \varphi, t) = \omega t + m\varphi + \int^\theta K \, d\theta'.$$

The disturbances from a given point source would propagate along the characteristics, or rays, of the equation (1). These rays can be characterized by a parameter, σ say, which determines the ray arc length from the source. It is therefore convenient to introduce the characteristic system of differential equations corresponding to (1) as follows:

$$\dot{\theta} = \frac{\partial G}{\partial K} = 2K \tag{III.2}$$

$$\dot{\varphi} = \frac{\partial G}{\partial m} = \frac{2m}{\sin^2\theta}\left(1 + \frac{\Omega^2}{\omega^2}\cos^2\theta\right) \tag{III.3}$$

$$\dot{t} = \frac{\partial G}{\partial \omega} = -2\left[\gamma^*\omega - \frac{m^2\Omega^2}{\omega^3}\cot^2\theta\right] \tag{III.4}$$

$$\dot{K} = -\frac{\partial G}{\partial \theta} = 2\left[m^2\left(1+\frac{\Omega^2}{\omega^2}\right) + \frac{1}{4}\right]\frac{\cos\theta}{\sin^3\theta} - \gamma^*\Omega^2\sin 2\theta \tag{III.5}$$

$$\dot{m} = -\frac{\partial G}{\partial \varphi} = 0 \tag{III.6}$$

$$\dot{\omega} = -\frac{\partial G}{\partial t} = 0 \tag{III.7}$$

where (•) means differentiation with respect to σ .

III.1. Ray Properties

The Eqs.(6),(7) simply imply that along any particular ray ω and m are constants. Thus waves with different values of ω and m will have different paths in the region $D(\omega, m)$. The parametric equations of A ray are defined by (2,3). The differential equation for A ray is obtained from these as

$$\frac{\partial\varphi}{d\theta} = \frac{m}{K\sin^2\theta}\left(1 + \frac{\Omega^2}{\omega^2}\cos^2\theta\right) \tag{III.8}$$

To integrate (8) in the general case demands some numerical work which is not necessary for just knowing the qualitative behavior of the rays. By inspection, keeping in mind the variation of K as function of θ , we can picture the form of a ray of given ω and m as in Fig. 3a.

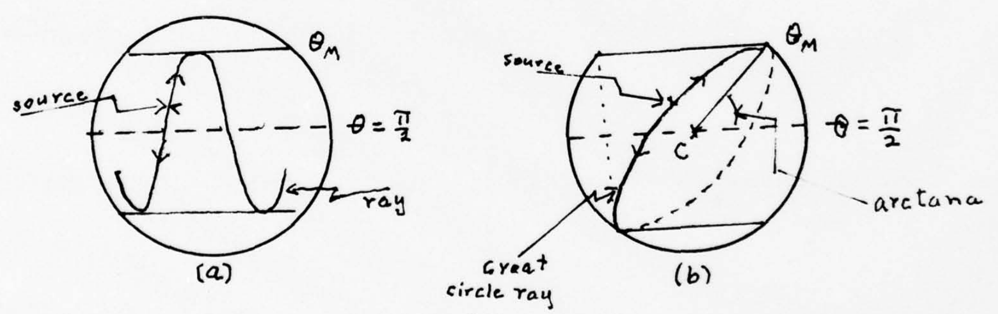

Fig.3

Just as a check we consider the simplest case in which ω and m are much larger than Ω so that the effect of rotation can be neglected. Then (8) becomes

$$\frac{d\varphi}{d\theta} = \frac{\pm 1}{\sin\theta\left(\frac{\gamma^*\omega^2}{m^2}\sin^2\theta - 1\right)^{1/2}}$$

with the solution

$$\begin{cases} \sin^2\theta\left(a^2\sin^2\varphi + 1\right) = 1 \\ \cos\theta = a\sin\theta\sin\varphi \ ; \quad a^2 = \dfrac{\gamma^*\omega^2}{m^2} - 1 \end{cases}$$

This is the equation of a great circle with the inclination angle with respect

to the equator equal to arctan a (see Fig.3b). The definition of the equator is completely arbitrary in this case (no preferred direction) and so is the orientation of the great circle. Of course this is to be expected in case of no rotation. In other simple cases the evaluation of (8) involves elliptic integrals.

III.2 Group Velocity

The energy of the disturbance, provided initially at the source, propagated along these rays with the group velocity. From (2, 3, 4) the θ and φ components of the angular group velocity are

$$C_{g\theta} = \frac{\dot{\theta}}{\dot{t}} = \frac{\partial \omega}{\partial K} = \frac{-K}{\gamma^* \omega + \frac{m^2 \Omega^2}{\omega^3} \cot^2 \theta} \qquad \text{(III.9)}$$

and

$$C_{g\varphi} = \frac{\dot{\varphi}}{\dot{t}} = \frac{\partial \omega}{\partial m} = \frac{-m\left(1 + \frac{\Omega^2}{\omega^2} \cos^2 \theta\right)}{\gamma^* \omega \sin^2 \theta + \frac{m^2 \Omega^2 \cos^2 \theta}{\omega^3}} \qquad \text{(III.10)}$$

It is easily seen that $|C_{g\theta}|$ decreases monotonically with θ from the maximum value at the equator, which is

$$\left|C^e_{g\theta}\right| = \frac{1}{\sqrt{\gamma^*}} \left[1 - \frac{m^2}{\gamma^* \omega^2}\right]^{1/2} \qquad \text{(III.11)}$$

to zero at $\theta = \theta_m$. Also $|C_{g\varphi}|$ has the opposite variation. Its minimum value, at the equator, is

$$\left|C^e_{g\varphi}\right| = \left|\frac{m}{\gamma^* \omega}\right| \qquad \text{(III.12)}$$

Also $|C_g| = \left|C^2_{g\theta} + C^2_{g\varphi}\right|^{1/2}$ has a maximum at the equator with the constant value $(\gamma^*)^{-1/2}$.

III.3 Phase Velocity

By definition the θ and φ components of the angular phase velocity are given by

$$C_{p\theta} = \frac{\omega K}{K^2 + m^2} \qquad \text{(III.13)}$$

$$C_{p\varphi} = \frac{\omega m}{K^2 + m^2} \qquad \text{(III.14)}$$

At the equator, where rotation has no effect, we expect the waves to propagate along the direction of the corresponding rays. That is, the group and phase velocity are identical except they are in opposite directions, as can be seen from their expressions. The fact that the phase velocity is constant, independent of ω and m, at the equator can be interpreted as in the case of shallow water waves.

The variation of $|C_{p\varphi}|$ is similar to $|C_{g\varphi}|$. As to $|C_{p\theta}|$ we have from (13)

$$\frac{\partial |c_{p\theta}|}{\partial \theta} = \frac{|\omega|\,(m^2-K^2)}{(K^2+m^2)^2} \cdot \frac{\partial |K|}{\partial \theta} .$$

where $\frac{\partial |K|}{\partial \theta}$ is positive with its value zero at $\theta = \frac{\pi}{2}$. Now $m^2 - K^2 = -(F - 2m^2)$. Therefore $\frac{\partial |C_{p\theta}|}{\partial \theta}$ has another zero if for given ω

$$2m^2 < F_{max} = \gamma^* \omega^2 .$$

In this case there exists θ_p, $\theta_M < \theta_p < \frac{\pi}{2}$, where $\frac{\partial |C_{p\theta}|}{\partial \theta}$ changes its sign from positive to negative. As m^2 increases the region of positive sign becomes narrower and narrower until it disappears for $m^2 = \gamma^* \omega^2$ (see Fig.4).

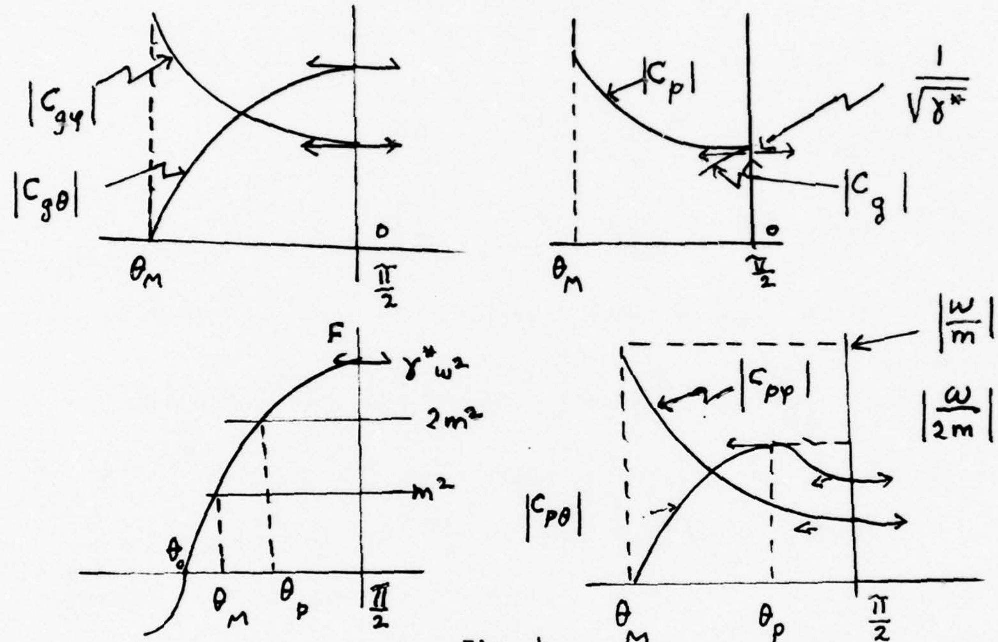

Fig. 4

In Fig.4 the value of $|C^e_{g\varphi}| = |C^e_{p\varphi}|$ is less than $|C^e_{g\theta}| = |C^e_{p\theta}|$ in the case $2m^2 < \gamma^* \omega^2$, otherwise it is the other way around.

III.4 Justification for Small Variation of Wave Amplitudes

Earlier we assumed that the variations of V, W and Θ with respect to θ are small so that the disturbances have a wave-like form. Now we try to justify that assumption in the region where the results should be reliable. This region is centered around the equator, i.e., θ close to $\frac{\pi}{2}$. To make the borders, where $\theta = \theta_M$ or $\theta = \theta_M + \frac{\pi}{2}$, far from the equator we assumed $m^2 \ll \omega^2$.This is not a constraint upon the following arguments; it just insures the existence of a good region of large width (see Fig.5.).

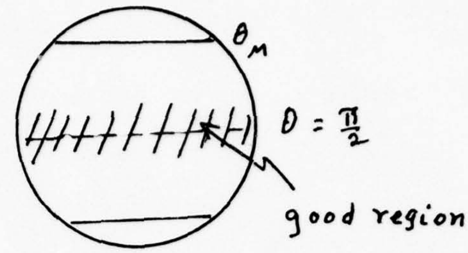

Fig.5

From (II.2-4) we can eliminate v' and w' to get the following equation for

$$\bar{\xi} = \Theta(\theta)e^{i\int^{\theta}(K+i\alpha)d\theta'}$$

where α is given by (II.6):

$$\frac{d^2\bar{\xi}}{d\theta} + \left[\left(1+\frac{2m\Omega}{\omega}\right)\cot\theta + \frac{2\Omega^2\sin\theta\cos\theta}{\omega^2+\Omega^2\cos^2\theta}\right]\frac{d\bar{\xi}}{d\theta}$$

$$+\left[\gamma^*(\omega^2+\Omega^2\cos^2\theta) - \frac{m^2}{\sin^2\theta} - \frac{m\Omega}{\omega} + \frac{2\Omega^3 m\cos^2\theta}{\omega(\omega^2+\Omega^2\cos^2\theta)}\right]\bar{\xi} = 0$$

This is similar to the Laplace tidal equation. In the good region defined above it reduces to

$$\frac{d^2\bar{\xi}}{d\theta^2} + \delta\frac{d\bar{\xi}}{d\theta} + (\gamma^*\omega^2 - m^2)\bar{\xi} \simeq 0 \ ; \ \delta \text{ is small.}$$

By differentiating $\bar{\xi}$ and neglecting small terms this becomes

$$\left[-K^2 + (\gamma^*\omega^2 - m^2)\right]\Theta + i\left[\left(\delta K + \frac{dK}{d\theta}\right)\Theta + 2K\frac{d\Theta}{d\theta}\right] = 0$$

Also in the good region $K^2 \simeq \gamma^*\omega^2 - m^2$. Therefore the real part is automatically almost zero. The imaginary part can be written as

$$-\frac{\Theta}{2}\left(\delta + \frac{1}{K}\frac{dK}{d\theta}\right) \simeq \frac{d\Theta}{d\theta}$$

But again in the good region $\left|\frac{1}{K}\frac{dK}{d\theta}\right| \ll |\delta|$; thus

$$\frac{d\Theta}{d\theta} = O(\delta\Theta).$$

Since Θ is the amplitude of small disturbances, we have therefore justified our assumption that the derivative of Θ is small compared to Θ .

Acknowledgments

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SLURRY DYNAMICS AND THE GEODYNAMO

Christopher L. Frenzen

Introduction

The most likely origin of the earth's magnetic field is a dynamo process in the conducting liquid core; the interaction of fluid motion, magnetic field, and electric current combine to generate an additional magnetic field capable of sustaining itself against ohmic and viscous losses. The maintenance of this field has never faltered (yet) and although frequent reversals in polarity (on a time scale of order several hundred thousand years) have been recorded by paleomagnetic measurements, these same observations suggest that the strength of the main field has changed little over the past three billion years. Numerous theories have arisen to explain the geodynamo and all must rely on some source of power capable of maintaining the motions necessary for dynamo action over most of geological time. The most likely energy sources are internal radioactive heating, latent heat released from the gradual cooling and solidification of the inner core, and gravitational energy supplied by the accretion of a dense inner core. Since the fluid motions generated by thermally driven and gravitationally driven convection are essentially the same, little distinction has been made between the two in models describing the generation of magnetic fields. However their thermal character is quite different.

Since convection is a very efficient process for transporting heat, a convecting thermal dynamo must transfer much of the total heat supplied to the core mantle boundary; however the high thermal conductivity of the liquid core greatly aids this transfer and subtracts from the thermal dynamo's necessary convective motions thereby lowering its efficiency. Estimates of the ohmic heat flux in the core, the observed values of heat flux at the earth's surface, and the distribution of radioactive sources in the crust lead to the concern that the postulated thermal energy sources may not be large enough to drive the inherently inefficient thermal dynamo. The possibility of gravitational energy released by accretion of the inner core, first proposed by Braginsky (1963), appears as a very promising alternative. Judging the effectiveness of a power source by the amount of energy it puts directly into fluid motion and by the inevitability of energy flow into this mode rather than conversion into heat leads the gravitational dynamo strong favor over thermally driven models.

Thermal Regimes

The core probably consists of a number of elements, but often it is simplified to a binary alloy composed of a heavy metal (perhaps iron with a fraction of

nickel) and a light non-metal (sulfur or silicon). A sample phase diagram appears below.

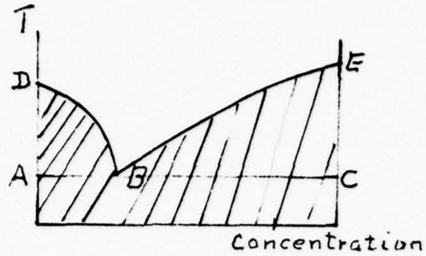

Fig.1

Temperature is plotted on the ordinate and concentration is taken as the abscissa. The unshaded regions represent homogeneous states while the shaded portions are regions of phase separation. In the liquid phase the components mix in all proportions. Above line ABC the mixed liquid phase is in equilibrium with one of the components. When the temperature of the liquid mixture decreases one or the other of the components freezes out depending on whether the concentration lies to the left or right of point B, the eutectic point. As the temperature decreases further, the liquid composition varies along curves DB and EB, and the liquid freezes completely at the eutectic point. DBE is known as the liquidus; above it only liquid exists. ABC is known as the solidus; below it only eutectic solid and one of the solid components exist, depending on the relative position of the eutectic point. At the latter, the liquid freezes totally into eutectic solid. The conducting liquid of the core is assumed more metallic than the eutectic; the solid freezing out of such a composition is more metallic and hence more dense than the surrounding liquid, and is capable of accreting on the inner core. (Loper (1978a)).

Following Loper (1978a) the various thermal regimes competible with the gravitationally powered dynamo will be described. Temperature gradients are taken with respect to pressure for thermodynamic convenience; these differ little from gradients taken with respect to depth because of the essentially hydrostatic pressure distribution in the earth. The conduction gradient, denoted by $\frac{\partial T_c}{\partial p}$ is defined as the change in temperature with respect to pressure necessary to remove heat in the absence of motion. The adiabatic temperature gradient

$$\left(\frac{\partial T_A}{\partial p}\right) = \left(\frac{\partial T}{\partial p}\right)_S = \left(\frac{\partial V}{\partial S}\right)_p = \frac{\partial T}{\rho C_p}$$

(Here V = volume, S = entropy, $C_p = T(\frac{\partial S}{\partial T})_p$, the specific heat at constant pressure, and $\alpha = \frac{1}{V}\left(\frac{\partial V}{\partial T}\right)_p$, the coefficient of thermal expansion) and the liquidus or melting temperature gradient

$\left(\frac{\partial T_L}{\partial p}\right) = \left(\frac{\Delta V}{\Delta S}\right)_p$ (where Δ denotes change upon melting)

are determined solely by the thermodynamic properties of the fluid. The conduction gradient, on the other hand, is determined by the temperature at the core - inner core and core-mantle boundaries. T and $\frac{\partial T}{\partial p}$ represent the actual temperature and temperature gradient respectively. Assume $\left(\frac{\partial TA}{\partial p}\right) > 0$, $\left(\frac{\partial T_L}{\partial p}\right) > 0$, and $\left(\frac{\partial T_c}{\partial p}\right) > 0$. (The latter implies a cooling core.)

The conventional thermal regime for the geodynamo results when

$$\left(\frac{\partial TA}{\partial p}\right) < \left(\frac{\partial T_c}{\partial p}\right) < \left(\frac{\partial T_L}{\partial p}\right)$$

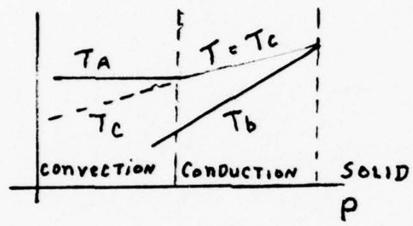

Fig. 2 (after Loper (1978a))

The core liquid, more metallic than the eutectic, freezes out a dense iron-rich solid capable of accreting on the inner core. The loss of a dense metallic solid to the inner core leaves the surrounding fluid compositionally buoyant. In the superadiabatic case $\left(\frac{\partial T_c}{\partial p}\right) > \left(\frac{\partial TA}{\partial p}\right)$ the fluid may convect thermally as well as compositionally. A thin conductive boundary layer will form near the inner core to transport away the latent heat released by the freezing of a dense metallic solid directly on to the core. However if $\left(\frac{\partial TA}{\partial p}\right) < \left(\frac{\partial T_L}{\partial p}\right) < \left(\frac{\partial T_c}{\partial p}\right)$

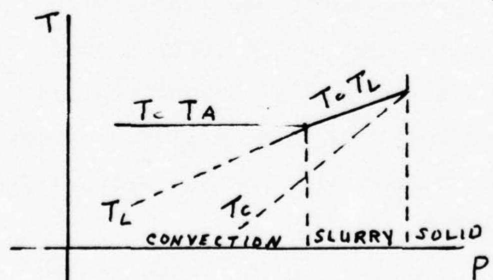

Fig. 3 (after Loper (1978a))

The fluid will still be compositionally and thermally buoyant but no conductive layer will exist for $\left(\frac{\partial T_L}{\partial p}\right) < \left(\frac{\partial T_c}{\partial p}\right)$ in such a layer implies it is frozen solid.

Here a slurry layer forms, a suspension of solid metallic particles in the fluid above the inner core. The slurry contains enough solid material so that the release of latent heat raises the "dry" solid free adiabatic gradient in the slurry layer. (The meteorological analogy is immediate.) The inner core grows through a "rain" of heavy iron-rich particles from the slurry layer above. (The concept of a slurry was first introduced by Busse (1972), Malkus (1973) and Elsasser (1972) to render the thermally stably stratified core of Higgins and Kennedy (1971) neutrally stable to the radial motions necessary for the geodynamo.) If

$$\left(\frac{\partial T_c}{\partial p}\right) < \left(\frac{\partial T_A}{\partial p}\right) < \left(\frac{\partial T_L}{\partial p}\right)$$

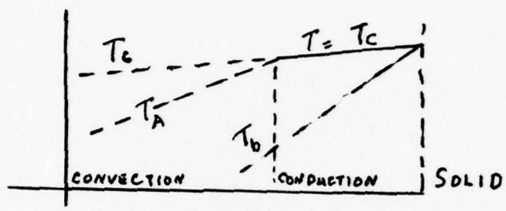

Fig. 4 (after Loper (1978a))

The core is subadiabatic and thermally stably stratified. Heat is removed by conduction and it may be, as Loper (1978a) assumes, that the fluid is compositionally buoyant enough to overcome the stable temperature gradient and convect gravitationally. Since the total heat transported arises from heat conduction down the conductive gradient as well as heat transport by convection, the buoyant convective motions may transport heat radially inward to make up for the excess heat conducted outward by the adiabatic temperature gradient.

A Simple Model of a Slurry

Consider a simple thermodynamic model of a slurry in equilibrium. While such a model is too simple for use in the earth's core, it gives a useful picture for the behavior of slurries. A slurry of constant composition formed from only one of a two-component mixture, e.g. a slurry of solid iron (Fe) particles formed from a non-eutectic mixture of iron and silicon (Fe- Si) can be described by the following 3 parameters:

α = mass fraction of slurry containing Si

α_L = mass fraction of the liquid phase containing only Si

γ = mass fraction of the slurry which is solid.

Then

$$\alpha = (1-\gamma)\alpha_L \tag{1}$$

for the slurry consists only of solid Fe. The free energy and free enthalpy G are

$$F = E - TS$$
$$G = E + PV - TS$$

The first law of thermodynamics for a mixture of two substances is

$$dE = Tds - p\,dV + \mu_1 dn_1 + \mu_2 dn_2$$

where n_1 and n_2 are the particle numbers of the two substances and μ_1, μ_2 are the chemical potentials. Hence for (1)

$$\begin{aligned} dF &= -P\,dV - S\,dT + (1-\gamma)\mu_L\, d\alpha_L + \mu_{SL}\, d\gamma \\ dG &= V\,dp - S\,dT + (1-\gamma)\mu_L\, d\alpha_L + \mu_{SL}\, d\gamma \end{aligned} \tag{2}$$

where $\mu_L = \mu_L^{i} - \mu_L^{Fe}$ = the chemical potential of the liquid phase

$\mu_{SL} = \mu_S^{Fe} - \left[(1-\alpha_L)\mu_L^{Fe} + \alpha_L \mu_L\right]$

= the chemical potential of the solid phase relative to the liquid phase. (Note μ_i^j = the chemical potential per unit mass of the ith phase of the jth substance).

Using the differential form of (1)

$$\begin{aligned} dF &= -P\,dV - S\,dT + \mu_L\, d\alpha + (\mu_{SL} + \alpha_L \mu_L)\, d\gamma \\ dG &= V\,dp - S\,dT + \mu_L\, d\alpha + (\mu_{SL} + \alpha_L \mu_L)\, d\gamma \end{aligned} \tag{3}$$

Let $U = \mu_{SL} + \alpha_L \mu_L = (\mu_S^{Fe} - \mu_L^{Fe})$ = the chemical potential above the equilibrium liquids value. In equilibrium $\mu_S^{Fe} = \mu_L^{Fe}$ and $U = 0$.

Gibb's phase rule for this system implies G is a function of three variables.

$$G = G(p, T, \alpha)$$

Expressions for $d\gamma$ and $d\alpha_L$ in terms of p, T and α are now sought.

Since $U(p, T, \alpha) = 0$

$$0 = dU = \left(\frac{\partial U}{\partial \alpha_L}\right)_{p,T,\gamma} d\alpha_L \left(\frac{\partial U}{\partial \gamma}\right)_{p,T,\alpha_L} d\gamma = \left(\frac{\partial U}{\partial p}\right)_{\alpha_L,\gamma,T} dp + \left(\frac{\partial U}{\partial T}\right)_{p,\alpha_L,\gamma} dT \tag{4}$$

Substituting for $d\alpha_L$ from (1) and solving for $d\gamma$, and then $d\alpha_L$

$$d\alpha_L = \frac{1}{A}\left[\frac{1}{\alpha_L}\left(\frac{\partial U}{\partial \gamma}\right)_{p,T,\alpha_L} d\alpha + \left(\frac{\partial U}{\partial p}\right)_{\alpha_L,T,\gamma} dp + \left(\frac{\partial U}{\partial T}\right)_{p,\alpha_L,\gamma} dT\right] \tag{5}$$

$$d\gamma = \frac{1-\gamma}{\alpha_L}\frac{1}{A}\left[\left(\frac{\gamma U}{\partial p}\right)_{\alpha_L,T,\gamma} dp + \left(\frac{\partial U}{\gamma T}\right)_{p,\alpha_L,\gamma} dT - \left(\left(\frac{\partial U}{\partial \alpha_L}\right)_{p,T,\gamma} \Big/ (1-\gamma)\right) d\alpha\right]$$

where $$A = \left(\frac{\partial U}{\partial \alpha_L}\right)_{p,T,\gamma} + \frac{1-\gamma}{\alpha_L}\left(\frac{\partial U}{\partial \gamma}\right)_{p,T,\alpha_L} \tag{6}$$

These differentials are valid within the slurry; outside the slurry $d\alpha = d\alpha_L$ and $d\gamma = 0$. Separate differentials inside and outside the slurry will lead to discontinuities in the fluid properties across the slurry edge. For example, consider the entropy S

$$dS = \left(\frac{\partial s}{\partial p}\right)_{T,\gamma,\alpha_L} dp + \frac{C_p}{T}\,dT + \left(\frac{\partial S}{\partial \alpha_L}\right)_{p,T,\gamma} d\alpha_L + \left(\frac{\partial s}{\partial \gamma}\right)_{T,\alpha_L,p} d\gamma \tag{7}$$

Substitutes $d\alpha_L$ and $d\gamma$ from (6) into (7)

$$dS = \left(\frac{\partial s}{\partial p}\right)_{SLURRY} dp + \frac{(C_p)_{SLURRY}}{T}\,dT + \left(\frac{\partial s}{\partial \alpha}\right)_{SLURRY} d\alpha$$

where

$$\left(\frac{\partial s}{\partial p}\right)_{SLURRY} = \left(\frac{\partial s}{\partial p}\right)_{T,\gamma,\alpha_L} - \frac{1}{A}\,\frac{1-\alpha}{\alpha_L}\left(\frac{\partial \upsilon}{\partial p}\right)_{\gamma,\alpha_L,T}\left(\frac{\partial \upsilon}{\partial T}\right)_{\gamma,\alpha_L,p}$$

$$(C_p)_{SLURRY} = C_p - T\,\frac{(1-\gamma)}{\alpha_L}\,\frac{1}{A}\left(\frac{\partial \upsilon}{\partial T}\right)^2_{\gamma,\alpha_L,p} \tag{8}$$

These discontinuities across the slurry edge are due to the change in specific volume (proportional to $\left(\frac{\partial \upsilon}{\partial p}\right)_{\gamma,\alpha_L,T}$) and the latent heat absorbed (proportional to $\left(\frac{\partial \upsilon}{\partial T}\right)_{\gamma,\alpha_L,p}$ due to the presence of solid matter in suspension.

The dry adiabatic temperature gradient is given by

$$\left(\frac{\partial T_A}{\partial p}\right) = \frac{-T\left(\frac{\partial s}{\partial p}\right)_{T,\gamma,\alpha_L}}{C_p}$$

(from use pf a Maxwell relation in (1)). By analogy, in the slurry we could expect the "wet" adiabatic gradient to be

$$\left(\frac{\partial T_A}{\partial p}\right)_{SLURRY} = \frac{-T\left(\frac{\partial s}{\partial p}\right)_{SLURRY}}{(C_p)_{SLURRY}} \tag{9}$$

These gradients differ because a fluid parcel moving into the slurry freezes out solid particles and releases enough latent heat to raise the temperature to tne liquidus. Normally the dry adiabatic gradient requires a temperature less than the liquidus. Hence in the slurry the liquidus should also be given by (9) as well. Therefore

$$\left(\frac{\partial T_L}{\partial p}\right) - \left(\frac{\partial T_A}{\partial p}\right) = \frac{T}{(C_p)_{SLURRY}}\left[\left(\frac{\partial s}{\partial p}\right)_{SLURRY} - \left(\frac{\partial s}{\partial p}\right)_{\gamma,\alpha_L,T}\frac{(C_p)_{SLURRY}}{C_p}\right]$$

$$= \frac{T}{(C_p)_{SLURRY}}\left(\frac{1-\gamma}{\alpha_L A}\right)\left(\frac{\partial \upsilon}{\partial T}\right)_{\gamma,\alpha_L,p}\left(\frac{\partial \upsilon}{\partial p}\right)_{\gamma,\alpha_L,T} - \frac{T\left(\frac{\partial s}{\partial p}\right)_{\gamma,\alpha_L,T}\left(\frac{\partial \upsilon}{\partial T}\right)_{\gamma,\alpha_L,p}}{C_p} \tag{10}$$

For $d\alpha = 0$ (α = constant in homogeneous slurry), (6) implies:

$$\frac{\partial \gamma}{\partial p} = \frac{1-\gamma}{\alpha_L}\,\frac{1}{A}\left[\left(\frac{\partial \upsilon}{\partial p}\right)_{\gamma,\alpha_L,T} + \left(\frac{\partial \upsilon}{\partial T}\right)_{\gamma,\alpha_L,p}\frac{dT}{dp}\right] = \frac{1-\gamma}{\alpha_L}\,\frac{1}{A}\left[\left(\frac{\partial \upsilon}{\partial p}\right)_{\gamma,\alpha_L,T} + \left(\frac{\partial \upsilon}{\partial T}\right)_{\gamma,\alpha_L,p}\left(\frac{-\left(\frac{\partial s}{\partial p}\right)_{SLURRY}T}{C_{p\,SLURRY}}\right)\right]$$

$$= \frac{1-\gamma}{\alpha_L}\,\frac{C_p}{A\,(C_p)_{SLURRY}}\left[\left(\frac{\partial \upsilon}{\partial p}\right)_{\gamma,\alpha_L,T} - \frac{T\left(\frac{\partial \upsilon}{\partial T}\right)_{\gamma,\alpha_L,p}\left(\frac{\partial s}{\partial p}\right)_{T,\gamma,\alpha_L}}{C_p}\right] \tag{11}$$

Combining (10) and (11)

$$\frac{d\gamma}{dp} = \frac{C_p}{\left(\frac{\partial V}{\partial T}\right)_{\gamma,\alpha_L,P} T} \left[\left(\frac{\partial T_L}{\partial P}\right) - \left(\frac{\partial T_A}{\partial P}\right) \right] \qquad (12)$$

If the liquidus is greater than the adiabatic gradient, the mass fraction of solid in the slurry increases with pressure (md depth) and the slurry rests at the bottom of the core. On the other hand, if the opposite is true, the slurry will occur in the outer core. Finally, by the Maxwell relation,

$$\left(\frac{\partial \mu}{\partial p}\right)_{T,N} = \left(\frac{\partial V}{\partial N}\right)_{P,T} = \sigma$$

where N is the particle number and σ is the specific volume. Hence

$$\left(\frac{\partial \mu_{SL}}{\partial p}\right)_{T,N} = \sigma_{SOLID} - \sigma_{liquid} = \frac{\rho_{liquid} - \rho_{SOLID}}{\rho_{liquid} - \rho_{SOLID}} \qquad (13)$$

since μ_{SL} = chemical potential of the solid relative to the liquid and $\rho = \frac{1}{\sigma}$. Therefore

$\left(\frac{\partial \mu_{SL}}{\partial p}\right)_{T,N} > 0$ implies particles are lighter than surrounding liquid.

$\left(\frac{\partial \mu_{SL}}{\partial p}\right)_{T,N} < 0$ implies particles are heavier than surrounding liquid.

For $\left(\frac{\partial \mu_{SL}}{\partial p}\right)_{T,N} < 0$ and $\frac{d\gamma}{dp} > 0$ particles can accrete on the core and compositional buoyancy will maintain the surrounding fluid in a homogeneous state.

However if $\left(\frac{\partial \mu_{SL}}{\partial p}\right)_{T,N} > 0$ or $\left(\frac{\partial \mu_{SL}}{\partial p}\right)_{T,N} < 0,\ \frac{\partial \gamma}{dp} < 0$

the slurry layer will gradually become stably stratified by sedimentation and thermally driven convection by latent heat release will be the only way of driving the dynamo.

The slurry layer imagined to drive the gravitationally powered dynamo differs from the one created to allow thermal convection in the thermally stably stratified core of Higgins and Kennedy (1971). The first slurry originates from a binary alloy on the metallic side of the eutectic, thus allowing compositional buoyancy; the second slurry consists of one metal and contains enough solid material to raise the dry adiabatic temperature gradient to the liquid and allow convective motions. Of course the slurry in the gravitationally-powered dynamo performs the latter function as well.

Loper (1978b) has concluded that the accreting core can supply as much as 1.76 10^{12} watts, enough to drive a large dynamo. He notes that the fraction of heavy material alloyed with light material in the frozen solid plays an

important role in gravitational energy release for once the composition in the outer core has evolved to the eutectic, further inner core growth will not result in separation of light and heavy material and the gravitationally-powered dynamo will cease to function.

Conclusion

Some possible thermal regimes of the gravitationally-powered dynamo and the qualitative behavior of a slurry have been examined. The direct non-thermal stirring by compositional convection lends the gravitationally-powered dynamo a large efficiency, defined as energy delivered to the kinetic mode over total energy delivered. Loper (1978b) has concluded that the efficiency may be as high as fifty percent. Further work along these lines would seek a suitable dynamical mechanism for the gravitationally-powered dynamo.

It is interesting to note that in the thermally stable stratified regime $\left(\frac{\partial T_C}{\partial P}\right) < \left(\frac{\partial T_A}{\partial P}\right) < \left(\frac{\partial T_L}{\partial P}\right)$ with an unstable compositional gradient double diffusion can occur. Any time gradients of two fluid properties with different molecular diffusivities are present and have opposing effects, double diffusion can take place. Since the thermal diffusivity is much greater than the molecular diffusivity of iron in an Fe - S or Fe - Si mixture, an iron-rich parcel displaced downward will lose its heat faster to its surroundings than its iron - it will be heavier and continue downward. Large scale "iron fingering" in the core coupled with the Alfven or MAC wave mechanism in hydromagnetic fluids could lead to collective instability and convection analogous to internal wave - salt finger instabilities (see The Collective Instability of Salt Fingers by J.Holyer in this same volume). Perhaps further work with double-diffusive experiments modelling conditions in the core can shed more light on dynamical mechanisms in the gravitationally-powered dynamo.

References

Braginsky, S.I. 1963 Structure of the F layer and reasons for convection in the earth's core. Poklady Akad.Nauk. SSSR, 149: 8.

Busse, F.H. 1972 Comment on the paper "The adiabatic gradient and the melting point gradient in the core of the earth" by G.Higgins and G.C.Kennedyl J.Geophys.Res. 77: 1589.

Elsasser, W.M. 1972 Thermal stratification md core convection, Int.Conf.Core-Mantle Interface, Trans.Am.Geophys.Union 53: 605.

Higgins, G.H. and G.C.Kennedy 1971 The adiabatic gradient and the melting point gradient in the core of the earth. J.Geophys.Res. 76: 1870.

Loper, D.E. 1978a Some thermal consequences of a gravitationally-powered dynamo. J.Geophys.Res., to appear.

Loper, D.E. (1978b) The gravitationally powered dynamo. Geophys.J.Roy.Ast.Soc., to appear.

Malkus, W.V.R. 1973 Convection at the melting point, a thermal history of the Earth's core. Geophys.Fluid Dyn. 4: 267.

STRANGE ATTRACTORS DUE TO FEEDBACK IN POTENTIAL SYSTEMS

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We consider systems of nonlinear ordinary differential equations of the form

$$\text{(i)} \quad \ddot{x} + \varepsilon\mu\dot{x} + V_x = 0$$

$$\text{(ii)} \quad \dot{\lambda} + \varepsilon(\lambda + g(x)) = 0$$

where x and λ are scalars and the "potential" V is a polynomial in x (a "cuspoid", (8)), and λ is one of its coefficients. The dependence of x on V, of V on λ, and of λ on x provides the "feedback" mechanism of the title. We have been particularly concerned with the systems

$$(1) \quad \begin{aligned} \ddot{x} &= -x^3 + \lambda x + \delta + \varepsilon\mu\dot{x} \\ \dot{\lambda} &= -\varepsilon(\lambda + \alpha(x^2 - a)), \end{aligned}$$

$$(2) \quad \begin{aligned} \ddot{x} &= x^2 - \lambda \\ \dot{\lambda} &= -\varepsilon\left(\lambda - \left(x^2 + ax + \frac{a^2}{2}\right)\right). \end{aligned}$$

The equations modelling a large number of physical systems can be put in this form. Examples include the Lorenz-Salzman model for Bénard convection (6), which can be transformed to Eq.(1), with $\delta = 0$; the Bullard-Howard-Malkus model for the geodynamo (7,9), involving the same equation; the Moore-Spiegel model for a Boussinesq fluid with a linear restoring force (1), which can also be transformed to Eq.(1), but with the roles of λ and δ interchanged in V, $\mu = 0$, and the function $g(x) = x^3 - x$ in the second ($\dot{\lambda}$) equation; and the generalized van der Pol oscillator of Rossler (10), which can be transformed to Eq.(2).

As we are interested in the assymptotic behavior (as $t \to \infty$) of solutions, we study the isolated invariant sets, particularly those which are locally stable (attracting), although exchange of stability associated with bifurcation (as ε varies) forces us also to consider sets which are not assymptotically stable (of saddle type, for example). The most mathematically interesting be-

havior occurs when ε is of moderate size, say .1 - .5 (the other parameters are between 1 and 10). For this reason, we have favored numerical and geometric over perturbation methods.

By "strange" attractor, we mean one which is not a differentiable manifold - so not a fixed point, limit cycle, or surface of any dimension. Such sets have been extensively studied since Smale's work on the horseshoe (11), which has the structure, locally, of the Cartesian product of a Cantor set, times itself, times a line interval. Preliminary numerical results seem to establish that a somewhat modified horseshoe is present in the system (2). Assuming that this is indeed the case, the methods of symbolic dynamics (2) establish the existence of infinitely many periodic orbits, collectively attracting though individually unstable, as well as an infinite number of other recurrent orbits in the same set. This situation, referred to with some justice as "chaotic", has been proposed as a model for turbulence (11).

The existence of a different type of chaotic invariant set for the system (1) with $\delta = 0$, attracting for $.1 < \varepsilon < .2$, has been established under some mild geometric assumptions (the "geometric Lorenz attractor", (3)), and also from numerical work giving apparently quite reliable estimates for an associated Poincaré first-return map (5). Moreover, the geometric Lorenz system has been shown (4,9) to be structurally unstable of codimension two - meaning that although a perturbation of these equations produces a flow which is not "topologically equivalent", in that it has a different (though similar) phase portrait, the resulting system is equivalent to a member of a two-parameter family of flows, obtained as follows: For ε between .1 and .2, the system (1) has three critical points, a node at (0, 0, A) and two foci at $\left(\pm\left(\frac{A}{1+a}\right)^{1/2}, 0, \frac{A}{1+a}\right)$, of unstable dimension 1,2,2 respectively. The unstable manifold of the node loops about the two foci; one forms a sequence with an 0 for each loop about the focus with negative x-coordinate, and a 1 for each loop about the focus whose x coordinate is positive. These "kneading sequences", which may be regarded as binary expansions for numbers between zero and one, completely determine the dynamics of the attracting set (9).

The methods of (2,5) were used to show that a strange attractor also exists for $\delta \neq 0$, and that one may manipulate the two kneading sequences by varying ε and δ, thus providing a concrete "unfolding" for the Lorenz system; that is, a particular realization for the two-parameter family of nearby systems.

The higher-order cuspoids will also, with appropriate parameters and feedback, exhibit strange attractors, since they "contain" the ones discussed above.

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References

1. Baker, Moore, Spiegel 1971 Aperiodic behavior of a nonlinear oscillator, Quart.J.Mech. and Appl.Math. 24: 391-422.
2. Bowen, 1978 Axiom A diffeomorphisms, Regional Conference Series in Mathematics 35, Am.Math.Sec.
3. Guckenheimer, 1976 A strange strange attractor, in The Hopf Bifurcation and Its Applications, ed. Marsden and McCracken, Appl.Math.Sci. 19: Springer-Berlag.
4. Guckenheimer, Structural Stability of Lorenz Attractors, preprint.
5. Kaplan and Yorke, Preturbulence: A Regime Observed in a Fluid Model of Lorenz, preprint.
6. Lorenz, 1963 Deterministic nonperiodic flow, J.Atm.Sci. 20: 130-141.
7. Malkus, 1972 EOS Trans.Am.Geophys. Union 54: 617.
8. Poston and Stewart 1978 Taylor expansions and catastrophes, Pitman.
9. Rand, 1978 Topological classification of Lorenz attractors, Proc.Camb. Phil.Soc. 83: 451-460.
10. Robbins 1977 A new approach to subcritical instability and turbulent Transition in a simple dynamo, Proc.Camb.Phil.Soc. 82: 309-325.
11. Rössler, 1977 Continuous chaos, in Synergetics: A Workshop, ed. Haken, Springer-Verlag: 184-197.
12. Ruelle and Takens, 1971 On the nature of turbulence, Comm.Math.Phys. 20: 167.
13. Smale, 1965 Diffeomorphisms with many periodic points, in Differential and combinatorial topology, ed. Cairns, Princeton Univ.Press: 63-80.

BAROTROPIC AND BAROCLINIC SOLITONS

Hisashi Hukuda

1. Introduction

Since Maxworthy and Redekopp (1976) presented the solitary Rossby wave theory as a dynamical model of the Great Red Spot observed in the Jovian atmosphere, there is no doubt that its theory has become "nouvelle vague" in the field of planetary atmospheric science. Redekopp (1977) developed the general

theory on the existence of solitary Rossby waves (planetary solitons) in the zonal shear flow on the continuously stratified atmosphere and showed that the evolution equation of solitary Rossby waves is the Korteweg de Vries (KdV) equation or the modified KdV (MKdV) equation depending on the distribution of density stratification. Redekopp and Weidman (1978) studied the interaction of two planetary solitons with a motivation to model the phase shift observed in the interactions between the Great Red Spot and other waves and showed that these interactions are described by a coupled KdV equation.

On the other hand, there is no evidence, so far, that solitary Rossby waves may really exist in the ocean. However, it seems to be not so unusual that we expect the existence of these waves and their interactions also in the ocean.

The solution for a single mode of solitary Rossby waves in a two-layer system was shown in Hukuda (1978). We attempt, in this report, to extend the same theme to the problem of interactions between two modes after Redekopp and Weidman (1978). The motivation for this problem is also related to what happens in the interaction of barotropic and baroclinic modes, the theme unexplored up to date and with an importance in understanding the large scale energy exchange between upper and deeper fluids in the ocean.

2. The Derivation of a Coupled Evolution Equation

The basic equation is a two-layer version of the conservation of potential vorticity. It is written below in a nondimensional form.

$$\left[\partial_t + U_n \partial_x + \epsilon(\varphi_{nx}\partial_y - \varphi_{ny}\partial_x)\right] \times \left[\mu^2 \varphi_{nxx} + \epsilon_n F(\varphi_2 - \varphi_1)\right] + \varphi_{nx}\left[-U_n'' + \epsilon_n F(U_1 - U_2) + \beta\right] = 0 \qquad (2.1)$$

with boundary conditions:

$$\varphi_{nx} = 0 \quad \text{at} \quad y = 0, 1.$$

$$\varphi_n \to 0 \quad \text{at} \quad |x| \to \infty.$$

where $n = 1, 2$ and $\epsilon_1 = 1$, $\epsilon_2 = -1$. The subscript $n = 1, 2$ refers the quantities of upper and lower layers, respectively. The nondimensional parameters have the definition:

$$F = \frac{4f^2L^2}{\frac{\Delta\rho}{\rho} g H^2} \quad \text{(the internal rotational Froude number)}$$

$$\beta = \frac{\beta^* L^2}{U} \quad \text{(the so-called B parameter)}$$

Equation (2.1) is derived from Pedlosky's (1970) by the transformation

$$\begin{cases} \Psi_n = -\int U_n(y)\,dy + \epsilon\,\varphi_n(x,y,t) \\ x \longrightarrow \mu x,\ t \longrightarrow \mu t, \end{cases}$$

where $U_n(y)$ denotes a basic flow, φ_n a perturbation stream function and small parameters $\epsilon, \mu (\ll 1)$ measures of disturbance amplitude and length scale, respectively.

The asymptotic solution of (2.1) may be written, after Benney (1966), in the form

$$\varphi_n = \varphi_n^{(0,0)} + \epsilon\,\varphi_n^{(1,0)} + \mu^2 \varphi_n^{(0,1)} + \ldots \tag{2.2}$$

By substitution of (2.2) into (2.1), we have the lowest order problem.

$$(\partial_t + U_n \partial_x)\left[\varphi_{n\,yy}^{(0,0)} + \epsilon_n F(\varphi_2^{(0,0)} - \varphi_1^{(0,0)})\right] + \varphi_{nx}^{(0,0)}\left[-U_n'' + \epsilon_n F(U_1 - U_2) + \beta\right] = 0 \tag{2.3}$$

Equation (2.3) has generally modal solutions 2 x ∞ for an arbitrary travelling wave. We study the interaction of two modes by writing the solution of (2.3) in the form

$$\varphi_n^{(0,0)} = A_1(x,t) f_{n1}(y) + A_2(x,t) f_{n2}(y) \tag{2.4}$$

Suppose that

$$\left.\begin{aligned} A_{1,t} &= -C_1 A_{1,x} + O(\epsilon, \mu^2) \\ A_{2,t} &= -C_2 A_{2,x} + O(\epsilon, \mu^2) \end{aligned}\right\} \tag{2.5}$$

where C_1 and C_2 ($C_1 \neq C_2$) represent the phase speeds of different long wave modes.

Then the modal functions must satisfy

$$\begin{gathered} f_{nj}'' + \epsilon_n F(f_{2j} - f_{1j}) + \frac{P_n}{U_n - C_i} f_{nj} = 0 \\ f_{nj} = 0 \text{ at } y = 0, 1. \quad (i = 1,2;\ j = 1,2) \end{gathered} \tag{2.6}$$

where $\quad P_n = U_n'' + \epsilon_n F(U_1 - U_2) + \beta$

For later convenience, we introduce the linear operator:

$$D_{ni}[f_{nj}] \equiv (U_n - C_i)\left[f_{nj}'' + \epsilon_n F(f_{2j} - f_{1j}) + \frac{P_n}{U_n - C_i} f_{nj}\right].$$

Before proceeding to the next order, we note the orthogonality condition implied by (2.6)

$$\int_0^1 \sum_{n=1}^{2} \frac{f_{n1}}{U_n - C_1} \frac{f_{n2}}{U_n - C_2} P_n\,dy = 0. \tag{2.7}$$

The problems of $O(\epsilon)$ and $O(\mu^2)$ are:

$$(\partial_t + U_n \partial_x)\left[\varphi_{nyy}^{(1,0)} + \epsilon_n F(\varphi_2^{(1,0)} - \varphi_1^{(1,0)})\right] + P_n \varphi_{nx}^{(1,0)}$$

$$= -(\varphi_{nx}^{(0,0)}\partial_y - \varphi_{ny}^{(0,0)}\partial_x)\left[\varphi_{nyy}^{(0,0)} + \epsilon_n F(\varphi_2^{(0,0)} - \varphi_1^{(0,0)})\right] \tag{2.8}$$

$$-K_1(A_i A_j)\left[f_{n1}'' + \epsilon_n F(f_{21} - f_{11})\right]$$

$$-K_2(A_i A_j)\left[f_{n2}'' + \epsilon_n F(f_{22} - f_{12})\right]$$

$$(\partial_t + U_n \partial_x)\left[\varphi_{nyy}^{(0,1)} + \epsilon_n F(\varphi_2^{(0,1)} - \varphi_1^{(0,1)})\right] + P_n \varphi_{nx}^{(0,1)}$$

$$= -(\partial_t + U_n \partial_x)\varphi_{nxx}^{(0,0)} \tag{2.9}$$

$$-L_1(A_1)\left[f_{n1}'' + \epsilon_n F(f_{21} - f_{11})\right]$$

$$-L_2(A_2)\left[f_{n2}'' + \epsilon_n F(f_{22} - f_{12})\right]$$

In the above, Eq.(2.5) was written:

$$\left.\begin{aligned} A_{1,t} + C_1 A_{1,x} &= \epsilon K_1(A_i A_j) + \mu^2 L_1(A_1) + O(\epsilon^2) \\ A_{2,t} + C_2 A_{2,x} &= \epsilon K_2(A_i A_j) + \mu^2 L_2(A_2) + O(\epsilon^2) \end{aligned}\right\} \tag{2.10}$$

This means that amplitudes of linear waves must be modified by the effect of nonlinearity and dispersion (Benney (1966)).

K_i and L_i generally have the form:

$$K_i(A_i A_j) = \gamma_{i1} A_1 A_{1,x} + \gamma_{i2} A_2 A_{2,x} + \nu_{i1} A_2 A_{1,x} + \nu_{i2} A_1 A_{2,x}$$
$$L_i(A_i) = S_i A_{i,xxx} \quad (i = 1, 2).$$

By inspection, the solutions of Eqs.(2.8) and (2.9) are

$$\left.\begin{aligned} \varphi_n^{(1,0)} &= (A_1^2/2)\, g_{n1} + (A_2^2/2)\, g_{n2} + A_1 A_2\, g_{n3} \\ \varphi_n^{(0,1)} &= A_{1,xx}\, h_{n1} + A_{2,xx}\, h_{n2} \end{aligned}\right\} \tag{2.11}$$

The modal functions must satisfy the sequence of inhomogeneous problems:

$$\left.\begin{aligned} D_{n1}[g_{n1}] &= \left(\frac{P_n}{U_n - C_1}\right)' f_{n1}^2 + r_{11}\frac{P_n f_{n1}}{U_n - C_1} + r_{21}\frac{P_n f_{n2}}{U_n - C_2} \\ D_{n2}[g_{n2}] &= \left(\frac{P_n}{U_n - C_2}\right)' f_{n2}^2 + r_{12}\frac{P_n f_{n1}}{U_n - C_1} + r_{22}\frac{P_n f_{n2}}{U_n - C_2} \\ D_{n1}[g_{n3}] &= \left\{\left(\frac{P_n f_{n2}}{U_n - C_2}\right)' - \frac{P_n f_{n2}'}{U_n - C_1}\right\} f_{n1} + \nu_{11}\frac{P_n f_{n1}}{U_n - C_1} + \nu_{21}\frac{P_n f_{n2}}{U_n C_2} \\ D_{n2}[g_{n3}] &= \left\{\left(\frac{P_n f_{n1}}{U_n - C_1}\right)' - \frac{P_n f_{n1}'}{U_n - C_2}\right\} f_{n2} + \nu_{12}\frac{P_n f_{n1}}{U_n - C_1} + \nu_{22}\frac{P_n f_{n2}}{U_n C_2} \end{aligned}\right\} \tag{2.12}$$

$$\left.\begin{aligned} D_{n1}[h_{n1}] &= S_1 \frac{P_n f_{n1}}{U_n - C_1} - (U_n - C_1) f_{n1} \\ D_{n2}[h_{n2}] &= S_2 \frac{P_n f_{n2}}{U_n - C_2} - (U_n - C_2) f_{n2} \end{aligned}\right\} \qquad (2.12)$$

One can show by using the solvability condition and the orthogonality condition (2.8) that

$$\left.\begin{aligned} \gamma_{11} &= \frac{-\int_0^1 \sum_{n=1}^{2} \frac{f_{n1}^3}{U_n - C_1} \left(\frac{P_n}{U_n - C_1}\right)' dy}{B_1} \\ \gamma_{22} &= \frac{-\int_0^1 \sum_{n=1}^{2} \frac{f_{n2}^3}{U_n - C_2} \left(\frac{P_n}{U_n - C_2}\right)' dy}{B_2} \\ \nu_{11} &= \frac{-\int_0^1 \sum_{n=1}^{2} \frac{f_{n1}^2}{U_n - C_1} \left\{\left(\frac{f_{n2} P_n}{U_n - C_2}\right)' - \frac{f_{n2}' P_n}{U_n - C_1}\right\} dy}{B_1} \\ \nu_{22} &= \frac{-\int_0^1 \sum_{n=1}^{2} \frac{f_{n2}^2}{U_n - C_2} \left\{\left(\frac{f_{n1} P_n}{U_n - C_1}\right)' - \frac{f_{n1}' P_n}{U_n - C_2}\right\} dy}{B_2} \end{aligned}\right\} \qquad (2.13)$$

where

$$Bi = \int_0^1 \sum_{n=1}^{2} \frac{f_{ni}^2 P_n}{(U_n - C_i)^2} dy \qquad (i = 1, 2)$$

These are necessary conditions for the solutions of inhomogeneous problems (2.12) to exist. The above process to determine the coefficients also shows that the terms including γ_{ij}, ν_{ij} $(i \neq j)$ are redundant. Thus, we can set $\gamma_{ij} = \nu_{ij} = 0$ $(i \neq j)$, $\gamma_{ii} = \gamma_i$, $\nu_{ij} = \nu_i$ without the loss of generality.

Now, to this order, the wave amplitude equations take the form;

$$\left.\begin{aligned} A_{1,t} + C_1 A_{1,x} &= \epsilon(\gamma_1 A_1 A_{1,x} + \nu_1 A_2 A_{1,x}) + \mu^2 S_1 A_{1,xxx} \\ A_{2,t} + C_2 A_{2,x} &= \epsilon(\gamma_2 A_2 A_{2,x} + \nu_2 A_1 A_{2,x}) + \mu^2 S_2 A_{2,xxx} \end{aligned}\right\} \qquad (2.14)$$

Equation (2.14) is a coupled KdV equation, if one takes $\epsilon = \mu^2$, which was derived by Tedekopp and Weidman (1978) in order to study the interactions between different long wave modes. However, as was shown in Hukuda (1978), a special situation yields vanishing coefficients in the nonlinear terms of the KdV equation. This occurs when one considers a purely baroclinic mode in the vertically uniform mean shear. In this case, one has the eigenfunction $f_{1i} = -f_{2i}$ for a baroclinic mode. A simple examination of the coefficients

(2.13) shows $r_1 = r_2 = \nu_1 = \nu_2 = 0$ for this mode. Therefore, Eq.(2.14) cannot describe the interactions of baroclinic modes in this special but interesting case. One should proceed to the next order ($O(\epsilon^2)$) for the completeness of soliton morphology.

When proceeding to the $O(\epsilon^2)$, we have

$$(\partial_t + U_n \partial_x)\left[\varphi_{nyy}^{(2,0)} + \epsilon_n F(\varphi_2^{(2,0)} - \varphi_1^{(2,0)})\right] + P_n \varphi_{nx}^{(2,0)}$$

$$= -(\varphi_{nx}^{(0,0)}\partial y - \varphi_{ny}^{(0,0)}\partial x)\left[\varphi_{nyy}^{(1,0)} + \epsilon_n F(\varphi_2^{(1,0)} - \varphi_1^{(1,0)})\right] \tag{2.15}$$

$$- (\varphi_{nx}^{(1,0)}\partial y - \varphi_{ny}^{(1,0)}\partial x)\left[\varphi_{nyy}^{(0,0)} + \epsilon_n F(\varphi_2^{(0,0)} - \varphi_1^{(0,0)})\right]$$

$$- M_1(A_i A_j A_k)\left[f_{n1}'' + \epsilon_n F(f_{21} - f_{11})\right]$$

$$- M_2(A_i A_j A_k)\left[f_{n2}'' + \epsilon_n F(f_{22} - f_{12})\right]$$

where Eq.(2.10) was written

$$A_{1,t} + C_1 A_{1x} = \epsilon K_1(A_i A_j) + \mu^2 L_1(A_1) + \epsilon^2 M_1(A_i A_j A_k) + O(\epsilon\mu^2, \mu^4)$$

$$A_{2,t} + C_2 A_{2x} = \epsilon K_2(A_i A_j) + \mu^2 L_2(A_2) + \epsilon^2 M_2(A_i A_j A_k) + O(t\mu^2, \mu^4)$$

with a general form of $M_i(A_i A_j A_k)$

$$M_i(A_i A_j A_k) = \alpha_{i1} A_1^2 A_{1,x} + \alpha_{i2} A_2^2 A_{2,x} + \chi_{i1} A_2^2 A_{1x} + \chi_{i2} A_1^2 A_{2,x}$$
$$+ \omega_{i1} A_1 A_2 A_{1,x} + \omega_{i2} A_1 A_2 A_{2,x} \qquad (i = 1,2)$$

The solution of Eq.(2.15) is

$$\varphi_n^{(2,0)} = (A_1^3/3)\chi_{n1} + (A_2^3/3)\lambda_{n2} + A_1^2 A_2 \lambda_{n3} + A_1 A_2^2 \chi_{n4} \tag{2.16}$$

By substituting (2.16) into (2.15) and repeating the same process as before, one can show that the terms including $\alpha_{ij}, \chi_{ij}, \omega_{ij}$ $(i \neq j)$ are redundant.

Thus we obtain the sequence of inhomogeneous equations which χ_{ni} $(i=1,4)$ must satisfy.

$$\left.\begin{aligned}
D_{n1}[\chi_{n1}] &= \frac{3}{2} f_{n1} g_{n1}\left(\frac{P_n}{U_n - C_1}\right)' - \frac{1}{2} f_{n1}^3 \left\{\frac{1}{U_n - C_1}\left(\frac{P_n}{U_n - C_1}\right)'\right\}' + \alpha_1 \frac{P_n f_{n1}}{U_n - C_1} \\
D_{n2}[\chi_{n2}] &= \frac{3}{2} f_{n2} g_{n2}\left(\frac{P_n}{U_n - C_2}\right)' - \frac{1}{2} f_{n2}^3 \left\{\frac{1}{U_n - C_2}\left(\frac{P_n}{U_n - C_2}\right)'\right\}' + \alpha_2 \frac{P_n f_{n2}}{U_n - C_2}
\end{aligned}\right\} \tag{2.17}$$

$$
\left.
\begin{aligned}
D_{n1}[\chi_{n3}] &= f_{n1} g_{n3}\left(\frac{P_n}{U_n-C_1}\right)' + \frac{1}{2}\left[g_{n1}\left(\frac{P_n f_{n2}}{U_n-C_2}\right)' - f_{n1}^{3}\left\{\frac{1}{U_n-C_1}\times\left(\frac{P_n f_{n2}}{U_n-C_2}\right)' - \frac{P_n f_{n2}'}{(U_n-C_1)^2}\right\} + \omega_1 \frac{P_n f_{n1}}{U_n-C_1}\right] \\
D_{n2}[\chi_{n3}] &= \frac{1}{2} f_{n2}\left\{\left(\frac{P_n g_{n1}}{U_n-C_1}\right)' - \frac{P_n g_{n1}'}{U_n-C_2}\right\} + f_{n1} g_{n3}\left(\frac{P_n}{U_n-C_1}\right)' - \frac{1}{2} f_{n1}^{2} f_{n2} \times \\
&\quad \left\{\frac{1}{U_n-C_1}\left(\frac{P_n}{U_n-C_1}\right)'\right\}' + \frac{f_{n1} f_{n1}'}{U_n-C_1}\left\{f_{n2} P_n\left(\frac{1}{U_n-C_2} - \frac{1}{U_n-C_1}\right)\right\}' + K_2 \frac{P_n f_{n2}}{U_n-C_2} \\
D_{n1}[\chi_{n4}] &= \frac{f_{n1}}{2}\left\{\left(\frac{P_n g_{n2}}{U_n-C_2}\right)' - \frac{P_n g_{n2}'}{U_n-C_1}\right\} + f_{n2} g_{n3}\left(\frac{P_n}{U_n-C_2}\right) - \frac{1}{2} f_{n1} f_{n2}^{2} \times \\
&\quad \left\{\frac{1}{U_n-C_2}\left(\frac{P_n}{U_n-C_2}\right)'\right\}' + \frac{f_{n2} f_{n2}'}{U_n-C_2}\left\{\left(\frac{1}{U_n-C_1} - \frac{1}{U_n-C_2}\right) P_n f_{n1}\right\} + K_1 \frac{P_n f_{ni}}{U_n-C_1} \\
D_{n2}[\chi_{n4}] &= f_{n2} g_{n3}\left(\frac{P_n}{U_n-C_2}\right)' + \frac{1}{2}\left[g_{n2}\left(\frac{P_n f_{n1}}{U_n-C_1}\right)' - f_{n2}^{3}\left\{\frac{1}{U_n-C_2}\left(\frac{P_n f_{n1}}{U_n-C_1}\right)'\right.\right. \\
&\quad \left.\left. - \frac{P_n f_{n1}'}{(U_n-C_2)^2}\right\} + \omega_2 \frac{P_n f_{n2}}{U_n-C_2}\right]
\end{aligned}
\right\} \tag{2.17}
$$

with the coefficients

$$
\left.
\begin{aligned}
\alpha_1 &= \frac{-\int_0^1 \sum_{n=1}^{2}\left[\frac{3}{2}\frac{f_{n1}^2 g_{n1}}{U_n-C_1}\left(\frac{P_n}{U_n-C_1}\right)' - \frac{1}{2}\frac{f_{n1}^4}{U_n-C_1}\left\{\frac{1}{U_n-C_1}\left(\frac{P_n}{U_n-C_1}\right)'\right\}\right] dy}{B_1} \\
\alpha_2 &= \frac{-\int_0^1 \sum_{n=1}^{2}\left[\frac{3}{2}\frac{f_{n1}^2 g_{n1}}{U_n-C_1}\left(\frac{P_n}{U_n-C_1}\right)' - \frac{1}{2}\frac{f_{n2}^4}{U_n-C_2}\left\{\frac{1}{U_n-C_2}\left(\frac{P_n}{U_n-C_2}\right)'\right\}\right] dy}{B_2} \\
K_1 &= \frac{\begin{aligned} &-\int_0^1 \sum_{n=1}^{2}\left[\frac{1}{2}\frac{f_{n1}^2}{U_n-C_1}\left\{\left(\frac{P_n g_{n2}}{U_n-C_2}\right)' - \frac{P_n g_{n2}}{U_n-C_1}\right\} + \frac{f_{n1} f_{n2} g_{n3}}{U_n-C_1}\left(\frac{P_n}{U_n-C_2}\right)'\right. \\ &\quad + \frac{1}{2}\frac{1}{U_n-C_1}\frac{1}{U_n-C_2}\left\{(f_{n1}^2)' f_{n2}^2\left(\frac{P_n}{U_n-C_2}\right)' + f_{n1}^2 (f_{n2}^2)'\left(\frac{P_n}{U_n-C_1}\right)'\right. \\ &\quad \left. + \frac{1}{2}(f_{n1}^2)'(f_{n2}^2)'\left(\frac{\cdot}{U_n-C_1} - \frac{P_n}{U_n-C_2}\right) - \frac{f_{n1}^2 f_{n2}^2 U_n'}{U_n-C_1}\left(\frac{P_n}{U_n-C_2}\right)\right] dy \end{aligned}}{B_1} \\
K_2 &= \left[\text{the subscripts } 1 \longrightarrow 2 \text{ or } 2 \longrightarrow 1 \text{ in the above}\right] \\
\omega_1 &\; \frac{\begin{aligned} &-\int_0^1 \sum_{n=1}^{2}\left[\frac{2 f_n^2 g_{n3}}{U_n-C_1}\left(\frac{P_n}{U_n-C_1}\right)' + \frac{f_{n1} g_{n1}}{U_n-C_1}\left(\frac{P_n f_{n2}}{U_n-C_2}\right)'\right. \\ &\quad \left. - \frac{f_{n1}^3}{U_n-C_1}\left\{\frac{1}{U_n-C_1}\left(\frac{P_n f_{n2}}{U_n-C_2}\right)' - \frac{P_u f_{n2}'}{(U_n-C_1)^2}\right\}'\right] dy \end{aligned}}{B_1} \\
\omega_2 &= \text{ the subscripts } 1 \longrightarrow 2 \text{ or } 2 \longrightarrow 1 \text{ in the above}
\end{aligned}
\right\} \tag{2.18}
$$

Now, to the order of ϵ^2, wave amplitude equations take the form:

$$
\begin{aligned}
A_{1,t} + C_1 A_{1,x} &= \epsilon(r_1 A_1 A_{1,x} + \nu_1 A_2 A_{1,x}) + \mu^2 S_1 A_{1,xxx} \\
&\quad + \epsilon^2(\alpha_1 A_1^2 A_{1,x} + \kappa_1 A_2^2 A_{1,x} + \omega_1 A_1 A_2 A_{1,x}) \\
A_{2,t} + C_2 A_{2,x} &= \epsilon(r_2 A_2 A_{2,x} + \nu_2 A_1 A_{2,x}) + \mu^2 S_2 A_{2,xxx} \\
&\quad + \epsilon^2(\alpha_2 A_2^2 A_{2,x} + \kappa_2 A_1^2 A_{2,x}) \omega_2 A_1 A_2 A_{2,x})
\end{aligned}
\tag{2.19}
$$

Note that the terms of $O(\epsilon^2)$ have a meaning only if the terms of $O(\epsilon)$ vanish (i.e. $r_i = \nu_i = 0$) and then the evolution equation appropriate to interactions of such modes is a coupled MkdV equation with the balance

3. The Classification of a Different Type of Interactions

Once given a coupled evolution equation, one can classify the different types of interactions between planetary solitons and constitute a possible feature of interactions. This is summarized in Tables 1 and 2.

TABLE 1

MEAN SHEAR	$U_1 = U_2$		
Type of interactions	B.T.- B.T.	B.T.- B.C.	B.C. - B.C.
Type of evolution equation	KdV - KdV	?	MkdV - MkdV

TABLE @

MEAN SHEAR	$U_1 \neq U_2$		
Type of interactions	B.T.- B.T.	B.T.- B.C.	B.C. - B.C.
Type of evolution equation	KdV - KdV		

B.T. = Barotropic Mode

B.C. = Baroclinic Mode

Table 1 shows the types of interactions and evolution equations in the absence of mean vertical shear. The interactions between barotropic modes obey a coupled KdV equation. On the other hand, the interaction between baroclinic modes is described by a coupled MkdV equation. In this special case, the interaction of barotropic and baroclinic modes is not possible, for both modes obey a different type of evolution equation.

However, when we consider a more general situation, that is, a baroclinic mean state, the above feature of interactions changes drastically.

Table 2 shows the interactive feature in such a case. There, the interactions between every pair of modes are described uniquely by a coupled KdV equation.

There are two types of interactions depending on a type of evolution equation. The solutions of these coupled evolution equations are obtained by following the theory of Oikawa and Yajima (1973).

These solutions and phase shift formulae are shown in the Appendix. Type I describes the interactions between every mode except for purely baroclinic modes and agrees with those of Redekopp and Weidman (1978) but for the difference of coefficients. Type II describes the interactions between purely baroclinic modes and differs from Type I in the form of phase variables. In the latter type of interaction, the phase of one mode is not only related to the phase of another mode, but also to its own phase. This is because the MKdV mode is strongly nonlinear compared with the KdV mode.

In order to substantiate these interactive features, we need solve the digenvalue problems and evaluate the values of coefficients γ_i, etc. However, this is beyond the scope of the present report.

4. Summary

A coupled evolution equation which describes the nonlinear interaction of solitary Rossby waves in a two-layer system was derived. Two types of interactions were classified depending whether these interactions obey a coupled KdV equation or a coupled MKdV equation. It was shown that, except for a special situation, a coupled KdV equation described appropriately the nonlinear interaction between solitary Rossby waves.

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References

Benney, D.J. 1966 Long nonlinear waves in fluid flows. J.Math. and Phys. 45: 52063.

Hukuda, H. 1978 Solitary Rossby waves in a two-layer system. Submitted to Tellus.

Maxworthy, T. and L.G.Redekopp 1976 A solitary wave theory of the Great Red Spot and other observed features in the Jovian atmosphere. Icarus,29: 261.

Oikawa, M. and N. Yajima 1973 Interactions of solitary waves - A perturbation approach to nonlinear systems. J.Phys.Soc.Japan, 34: 1093-1099

Pedlosky, J. 1970 Finite-amplitude baroclinic waves. J.Atmos.Sci. 27: 15-20.

Redekopp, L.G. 1977 On the theory of solitary Rossby waves. J.Sluid Mech., 82: 725-745.

Redekopp, L.G. and P.D.Weidman 1978 Solitary Rossby waves in zonal shear flows and their interactions. J.Atmos.Sci. 35: 790-804.

Appendix

The solutions of a coupled evolution equation and phase shift formulae are shown below:

TYPE I

Evolution equation:

$$A_{1,t} + C_1 A_{1,x} = \epsilon(r_1 A_1 A_{1,x} + \nu_1 A_2 A_{1,x} + S_1 A_{1,xxx})$$
$$A_{2,t} + C_2 A_{2,x} = \epsilon(r_2 A_2 A_{2,x} + \nu_2 A_1 A_{2,x} + S_2 A_{2,xxx})$$

Solution:

$$A_1 = A_0 \, sgn(r_1 S_1) \, sech^2 \zeta_1$$
$$A_2 = B_0 \, sgn(r_2 S_2) \, sech^2 \zeta_2$$

where

$$\zeta_1 = \epsilon^{1/2} \left| \frac{r_1 A_0}{12 S_1} \right|^{\frac{1}{2}} \left\{ X - V_1 T + \epsilon^{1/2} \frac{\nu_1 B_0}{C_1 - C_2} sgn(r_2 S_2) \left| \frac{12 S_2}{r_2 B_0} \right|^{1/2} \tanh \zeta_2 \right.$$
$$\zeta_2 = \epsilon^{1/2} \left| \frac{r_2 B_0}{12 S_2} \right|^{\frac{1}{2}} \left\{ X - V_2 T - \epsilon^{1/2} \frac{\nu_2 A_0}{C_1 - C_2} sgn(r_1 S_1) \left| \frac{12 S_1}{r_1 A_0} \right|^{1/2} \tanh \zeta_1 \right.$$

with

$$V_1 = C_1 - \epsilon \frac{|r_1| A_0}{3} sgn(S_1)$$
$$V_2 = C_2 - \epsilon \frac{|r_2| B_0}{3} sgn(S_2)$$

In the above, (X,T) represents the original coordinate

phase shift

$$\delta_1 = 2 \epsilon^{1/2} \frac{\nu_1 B_0}{C_1 - C_2} sgn(r_2 S_2) \left| \frac{12 S_2}{r_2 B_0} \right|^{\frac{1}{2}}$$
$$\delta_2 = 2 \epsilon^{1/2} \frac{\nu_2 A_0}{C_1 - C_2} sgn(r_1 S_1) \left| \frac{12 S_1}{r_1 A_0} \right|^{\frac{1}{2}}$$

THE COLLECTIVE INSTABILITY OF SALT FINGERS

Judith Holyer

1. Introduction

The salt finger mechanism was first discussed by Stommel *et al.* (1956), when it was considered an oceanographic curiosity of little practical or scientific importance. Since this time the subject of thermohaline convection has been studied both theoretically and experimentally by a number of authors and it is now believed to be a feature of major importance in transport processes in the ocean. Williams (1974) has observed salt fingers in the ocean thermocline and it is thought that the step structure in the thermocline is maintained by double diffusive processes.

Salt fingers can be formed when a layer of hot, salty fluid lies above a layer of cold, fresh fluid of greater density. It is possible to get a physical understanding of this instability by considering a blob of cold, fresh fluid in equilibrium with its surroundings. If we lift this blob of fluid it will move into a hotter, saltier environment. Since the thermal diffusivity is much larger than the salt diffusivity, the blob will first come into thermal equilibrium with its surroundings. However, it will still be fresher, and hence lighter, and so it will continue to rise. By an equivalent argumant, if a blob of hot, salty fluid is moved down in the fluid it will continue to fall.

The problem we wish to consider here is the stability of these salt fingers to large wavelength internal wave perturbations. Stern (1969, 1975) first studied this instability, which is known as the collective instability of salt fingers. This is because energy is fed from the groups of small scale fingers to the large scale wave motion. In his study Stern (1969, 1975) does not explicity investigate the coupling between the small scale and the large scale motions. He takes the averaged momentum, heat and salt equations and then relates each of the terms to the large scale. In order to do this he assumes that the Reynolds stress is negligible and that the salt fingers are rotated by the internal wave, but that the magnitude of the fluxes associated with them remains unaltered. He then finds that the fingers are unstable if

$$\frac{F_S - F_T}{\nu(\alpha T_z - \beta S_z)} > 1$$

where F_T and F_S are the heat and salt fluxes of the salt fingers. ν is the kinematic viscosity of the fluid and αT_z and β_S are the heat and salt gradients in the fluid.

In the work presented here we study the collective instability, but we explicitly consider the coupling between the salt fingers and the large scale motion. Including all possible couplings, we find there is instability if

$$\frac{(F_S - F_T)}{\nu(\alpha T_z - \beta S_z)} > \frac{1}{3}$$

Thus, we find that Stern's assumptions were not correct, although the form of the stability criterion is the same.

2. The Salt Finger Solution

Suppose we have an unbounded region of fluid which has a stable linear temperature gradient, T_z , and an unstable linear salt gradient, S_z , with the overall density statically stable, i.e. increasing with depth. The coordinate vertically upwards shall be taken as z and the horizontal coordinate shall be x. We shall consider only two-dimensional motions, so we can define a stream-function, ψ , by

$$u = -\frac{\partial \psi}{\partial z} \; ; \quad w = \frac{\partial \psi}{\partial x} , \tag{2.1}$$

where u is the horizontal velocity and w is the vertical velocity in the fluid. The temperature field, T' , and the salinity field, S' , will be given by

$$\begin{aligned} T' &= T_z z + T(x,z,t) \\ S' &= S_z z + S(x,z,t) \end{aligned} \tag{2.2}$$

The density field will be given by

$$\rho = \rho_0 (1 - (\alpha T_z - \beta S_z) - (\alpha T - \beta S)) \tag{2.3}$$

The two-dimensional equations of motion can then be written as

$$\begin{aligned} &\frac{\partial}{\partial t} \nabla^2 \psi + J(\psi, \nabla^2 \psi) = \frac{\partial}{\partial x}\left(g(\alpha T - \beta S)\right) + \nu \nabla^4 \psi \\ &\frac{\partial T}{\partial t} + J(\psi, T) + T_z \frac{\partial \psi}{\partial x} = K_T \nabla^2 T \\ &\frac{\partial S}{\partial t} + J(\psi, S) + S_z \frac{\partial \psi}{\partial x} = K_S \nabla^2 S \end{aligned} \tag{2.4}$$

where $J(\phi, \psi) = \frac{\partial \phi}{\partial x}\frac{\partial \psi}{\partial z} - \frac{\partial \phi}{\partial z}\frac{\partial \psi}{\partial x}$, the Jacobian, and $\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial z^2}$. The thermal diffusivity is K_T and the salt diffusivity K_S .

We now look for a steady solution to the equations (2.4) which represents the motion in the salt fingers. We try a solution in which

$$\psi = -\hat{W}\ell\cos\frac{x}{\ell}$$
$$T = \hat{T}\sin\frac{x}{\ell} \qquad (2.5)$$
$$S = \hat{S}\sin\frac{x}{\ell}$$

where $\hat{W}$, $\hat{T}$ and $\hat{S}$ are constants.

Substituting (2.5) in (2.4) we find (2.5) as a solution provided

$$\hat{W} = \frac{\ell^2 g}{\nu}(\alpha\hat{T} - \beta\hat{S}) \qquad (2.6a)$$

and

$$\hat{T} = -\frac{\ell^2 T_z}{K_T}\hat{W} \; ; \; \hat{S} = -\frac{\ell^2 S_z}{K_S}\hat{W} \qquad (2.6b)$$

These imply

$$\ell^4 = \frac{\nu}{g\left(\frac{\beta S_z}{K_S} - \frac{\alpha T_z}{K_T}\right)} \; . \qquad (2.7)$$

From (2.7) we see that we need

$$\frac{\beta S_z}{K_S} \cdot \frac{K_T}{\alpha T_z} > 1 \qquad (2.8)$$

for a solution to exist. Thus, as pointed out by Huppert and Manins (1973), (2.5) with the relationships (2.6) - (2.3) gives a steady solution to the fully nonlinear equations (2.4).

The heat flux in the fingers is given by

$$F_T = -\overline{\frac{\partial\psi}{\partial x}\alpha T}$$

where $(\overline{\quad})$ denotes a horizontal average over long distances.

Then $F_T = -\alpha\hat{W}\cdot\hat{T}\,\overline{\sin^2\frac{x}{\ell}} = -\frac{\alpha\hat{W}\hat{T}}{2}$ (2.9)

Similarly the salt flux

$$F_s = -\overline{\frac{\partial\Psi}{\partial x}\cdot\beta S} = -\frac{\beta\hat{W}\hat{S}}{2} \qquad (2.10)$$

So

$$\frac{F_s}{F_T} = \frac{\beta\hat{S}}{\alpha\hat{T}} \qquad (2.11)$$

By (2.6b) we then see

$$\frac{F_s}{F_T} = \frac{\beta S_z}{K_S}\cdot\frac{K_T}{\alpha T_z} \qquad (2.11)$$

There is a further useful relationship. If we multiply (2.6a) by $\hat{W}$ and use (2.9) and (2.10) we find

$$\frac{\hat{W}^2}{2\ell^2} = \frac{g}{\nu}\quad(F_s - F_T) \qquad (2.12)$$

This equation expresses the balance between the buoyancy flux and viscous dissipation flux. From (2.8) and (2.11) we see

$$\frac{F_S}{F_T} > 1.$$

$\frac{F_S}{F_T}$ is the ratio of the potential energy lost by the salt field to that gained by the temperature field, and so must be greater than one, since otherwise the system is gaining energy. From (2.12) we see that the energy that is not gained by the temperature field is dissipated by viscosity.

3. The Averaging Procedure

We now suppose

$$\begin{aligned}\Psi &= -\ell \hat{W} \cos \frac{x}{\ell} + \Psi(x, z, t) \\ T &= \hat{T} \sin \frac{x}{\ell} + T(x, z, t) \\ S &= \hat{S} \sin \frac{x}{\ell} + S(x, z, t)\end{aligned} \tag{3.1}$$

Substituting (3.1) into (2.4) and linearising gives

$$\left(\frac{\partial}{\partial t} - \nu \nabla^2\right) \nabla^2 \Psi - g\left(\alpha \frac{\partial T}{\partial x} - \beta \frac{\partial S}{\partial x}\right) = -\hat{W} \sin \frac{x}{\ell} \left(\frac{\partial}{\partial z} \nabla^2 \Psi + \frac{1}{\ell^2} \frac{\partial \Psi}{\partial z}\right) \tag{3.2a}$$

$$\left(\frac{\partial}{\partial t} - K_T \nabla^2\right) T + T_z \frac{\partial \Psi}{\partial x} = -\hat{W} \sin \frac{x}{\ell} \frac{\partial T}{\partial z} + \frac{\hat{T}}{\ell} \cos \frac{x}{\ell} \frac{\partial \Psi}{\partial z} \tag{3.2b}$$

$$\left(\frac{\partial}{\partial t} - K_S \nabla^2\right) S + S_z \frac{\partial \Psi}{\partial x} = -\hat{W} \sin \frac{x}{\ell} \frac{\partial S}{\partial z} + \frac{\hat{S}}{\ell} \cos \frac{x}{\ell} \frac{\partial \Psi}{\partial z} \tag{3.2c}$$

We now write the Eq.(3.2) in the operator form

$$Lu = Mu \tag{3.3}$$

where L is the operator on the left-hand side of (3.2), M is the "rapidly varying" operator on the right-hand side of (3.2), which is due to the salt finger field, and $u = (\Psi, T, S)^T$.

Since the coefficients in (3.3) are independent of z and t we can find solutions with

$$u = \mathcal{R}e\left(u(x) e^{(imz + i\omega t)}\right)$$

where m is a vertical wavenumber and ω is the wave frequency. Now we wish to consider perturbations, u, which vary over a horizontal length scale much larger than ℓ. However, the salt finger field forces motions which vary on a short length scale, ℓ. So we put

$$u(x) = u_m(x) + u_r(x) \tag{3.4}$$

where $u_m(x)$ is the mean part of the field, which varies over a length scale, $\frac{1}{k}$, and $u_r(x)$ is the rapidly varying part, which varies on a length scale, ℓ. For the mean field we shall look for wave solutions

$$u_m = \begin{pmatrix} \psi_m \\ T_m \\ S_m \end{pmatrix} = \mathbb{R}e \left(\begin{pmatrix} A \\ -iB \\ -iC \end{pmatrix} \exp(ikx + imz + i\omega t) \right) \tag{3.5}$$

The wavenumber of this wave is given by μ, where $\mu^2 = k^2 + m^2$. It is travelling at an angle θ to the vertical where

$$k = \mu \sin\theta \quad \text{and} \quad m = \mu\cos\theta .$$

The basis of our approximation will be $\mu << \frac{1}{\ell}$.

We define a new coordinate system (x' , z') with

$$\begin{aligned} x' &= x\sin\theta + z\cos\theta \\ z' &= -x\cos\theta + z\sin\theta \end{aligned} \tag{3.6}$$

In this system x' measures distance in the direction of propagation of the wave and z' measures distance perpendicular to the direction of propagation of the wave along the wave fronts. Then

$$u_m = \mathbb{R}e \left\{ \begin{pmatrix} A \\ -iB \\ -iC \end{pmatrix} e^{i\mu x' + i\omega t} \right\} \tag{3.7}$$

We define an averaging operator $\langle \; \rangle$ by

$$\langle f \rangle = \lim_{L\to\infty} \frac{1}{L} \int_0^L f \, dz' \tag{3.8}$$

Thus $\langle \; \rangle$ represents the average of a quantity along a wave front. So, from the definition, $\langle u_m \rangle = u_m$ and $\langle u_r \rangle = 0$. Since L is a linear operator with constant coefficients $\langle Lf \rangle = L\langle f \rangle$, so $\langle Lu_m \rangle = Lu_m$ and $\langle Lu_r \rangle = 0$. Also $\langle Mu_m \rangle = 0$ since the rapidly varying operator acting on a mean quantity will give a rapidly varying result. Thus, averaging (3.3), we find

$$Lu_m = \langle Mu_r \rangle \tag{3.9}$$

Subtracting (3.9) from (3.3) gives

$$Lu_r = Mu_m + Mu_r - \langle Mu_r \rangle \tag{3.10}$$

Now we expect $Mu_r - \langle Mu_r \rangle$ to be negligible compared with Mu_m since u_r is associated with the length scale ℓ and u_m is associated with the length scale $\frac{1}{\mu}$. Then, provided $\mu << \frac{1}{\ell}$, Eq.(3.10) becomes

$$Lu_r = Mu_m \tag{3.11}$$

This approximation is equivalent to the first order smoothing of kinematic MHD.

We verified the validity of this approximation when we had calculated u_m and u_r explicitly.

In order to solve (3.9) and (3.11) we use the relationship that

$$\mathbb{R}e(a)\,\mathbb{R}e(b) = \frac{1}{2}\left(\mathbb{R}(ab) + \mathbb{R}e(ab^*)\right) \tag{3.12}$$

where * denotes complex conjugate. Then from (3.11) we find

$$u_r = \mathbb{R}e\left\{\begin{pmatrix} iA_1 \\ B_1 \\ C_1 \end{pmatrix} e^{i(k-\frac{1}{l})x + imz + i\omega t} + \begin{pmatrix} -iD_1 \\ E_1 \\ F_1 \end{pmatrix} e^{i(k+\frac{1}{l})x + imz + i\omega t}\right\} \tag{3.13}$$

Substituting (3.13) and (3.5) in (3.11), using (2.6b) and (3.12), and equating coefficients, we find

$$-(i\omega + \nu\mu_-^2)\mu_-^2 A_1 + g\left(\frac{1}{l} - k\right)(\alpha B_1 - \beta C_1) = \frac{im\hat{W}}{2}\left(\mu^2 - \frac{1}{l^2}\right)A \tag{3.14a}$$

$$(i\omega + K_T\mu_-^2)B_1 + T_z\left(\frac{1}{l} - k\right)A_1 = -\frac{im\hat{W}}{2}\left(B + \frac{lT_z}{K_T}A\right) \tag{3.14b}$$

$$(i\omega + K_S\mu_-^2)C_1 + S_z\left(\frac{1}{l} - k\right)A_1 = -\frac{im\hat{W}}{2}\left(C + \frac{lS_z}{K_S}A\right) \tag{3.14c}$$

$$-(i\omega + \nu\mu_+^2)\mu_+^2 D_1 + g\left(\frac{1}{l} + k\right)(\alpha E_1 - \beta F_1) = \frac{im\hat{W}}{2}\left(\mu^2 - \frac{1}{l^2}\right)A \tag{3.15a}$$

$$(i\omega + K_T\mu_+^2)E_1 + T_z\left(\frac{1}{l} + k\right)D_1 = \frac{im\hat{W}}{2}\left(B - \frac{lT_z}{K_T}A\right) \tag{3.15b}$$

$$(i\omega + K_S\mu_+^2)F_1 + S_z\left(\frac{1}{l} + k\right)D_1 = \frac{im\hat{W}}{2}\left(C - \frac{lS_z}{K_S}A\right) \tag{3.15c}$$

where

$$\mu_+^2 = \left(\frac{1}{l} + k\right)^2 + m^2 \quad , \quad \mu_-^2 = \left(\frac{1}{l} - k\right)^2 + m^2$$

By substituting (3.13) and (3.5) into (3.9) we obtain

$$-(i\omega + \nu\mu^2)\mu^2 A - gk(\alpha B - \beta C) = \frac{im\hat{W}}{2}\left(\left(\mu_-^2 - \frac{1}{l^2}\right)A_1 + \left(\mu_+^2 - \frac{1}{l^2}\right)D_1\right) \tag{3.16a}$$

$$(i\omega + K_T\mu^2)B - T_z kA = \frac{im\hat{W}}{2}\left((-B_1 - E_1) - \frac{lT_z}{K_T}(D_1 - A_1)\right) \tag{3.16b}$$

$$(i\omega + K_S\mu^2)C - S_z kA = \frac{im\hat{W}}{2}\left((-C_1 - F_1) - \frac{lS_z}{K_S}(D_1 - A_1)\right) \tag{3.16c}$$

Now (3.14) - (3.16) constitute a set of nine linear, homogeneous, simultaneous equations in A, B, C, A_1, B_1, C_1, D_1, E_1, and F_1. Thus, in order for a solution to exist, the determinant of the coefficients of this set of equations must be zero. This determinant will give the dispersion relation

$\omega(k, m) = 0$. We can, in fact, somewhat simplify this procedure by solving (3.14) and (3.15) to give A_1, B_1, C_1, D_1, E_1, and F_1 in terms of A, B and C and then using this to substitute into (3.16). We do this in the next section.

We note here that it is possible to do the averaging using a different averaging operator and to obtain the same dispersion relation. If we define $\langle \ \rangle'$ as the horizontal x - average over distances $\mathcal{L}$ such that

$$\ell \ll \mathcal{L} \ll \frac{1}{k}$$

i.e. $\langle f \rangle' = \frac{1}{\mathcal{L}} \int_0^{\mathcal{L}} f \, dx$

then we obtain the equations $\qquad L u_m = \langle M u_r \rangle$

and $\qquad L u_r = M u_m$

provided $\mu \ll \frac{1}{\ell}$

and the dispersion relation remains identical.

4. The Stability Criterion

We solve (3.14) and (3.15) for A_1 to F_1 and substitute in (3.16), using $\mu \ll \frac{1}{\ell}$. We retain only the lowest order term in μ that multiplies each coefficient. Then

$$(i\omega + \nu\mu^2)\mu^2 A + gk(\alpha B - \beta C) = \frac{m^2 \hat{W}^2}{P}\Big(\alpha g k B\big(i\omega + \frac{K_S}{\ell^2}\big) - \beta g k C\big(i\omega + \frac{K_T}{\ell^2}\big) - AX\Big) \tag{4.1a}$$

$$(i\omega + K_T\mu^2) B - T_z k A = -\frac{m^2 \hat{W}^2}{2P}\Big(T_z g \beta C \frac{\ell^2}{K_T}\big(i\omega + \frac{2K_T}{\ell^2}\big) + B\Big(\big(i\omega + \frac{K_S}{\ell^2}\big)\big(i\omega + \frac{\nu}{\ell^2} - \ell^2 \frac{g\alpha T_z}{K_T}\big) - g\beta S_z\Big) + AY\Big) \tag{4.1b}$$

$$(i\omega + K_S\mu^2) C - S_z k A = -\frac{m^2 \hat{W}^2}{2P}\Big(-S_z g \alpha B \frac{\ell^2}{K_S}\big(i\omega + 2\frac{K_S}{\ell^2}\big) + C\Big(\big(i\omega + \frac{K_T}{\ell^2}\big)\big(i\omega + \frac{\nu}{\ell^2} + \ell^2 \frac{g\beta S_z}{K_S}\big) + g\alpha T_z\Big) + AZ\Big) \tag{4.1c}$$

where

$$P = \big(i\omega + \frac{\nu}{\ell^2}\big)\big(i\omega + \frac{K_T}{\ell^2}\big)\big(i\omega + \frac{K_S}{\ell^2}\big) + i\omega g(\alpha T_z - \beta S_z) - \frac{\nu K_S K_T}{\ell^6}$$

$$R = \big(i\omega + \frac{K_T}{\ell^2}\big)\big(i\omega + \frac{K_S}{\ell^2}\big) + \frac{i\omega\nu}{\ell^2} + g K_S K_T\Big(\frac{\beta S_z}{K_S^2} - \frac{\alpha T_z}{K_T^2}\Big)$$

$$S = \frac{2R}{\ell P}\Big(\frac{3\nu K_S K_T}{\ell^4} + \frac{2i\omega}{\ell^2}\big(K_S K_T + \nu(K_S + K_T)\big) - \omega^2(\nu + K_T + K_S)\Big) - 2\ell\big(\omega^2 + \frac{K_S K_T}{\ell^4}\big) + \frac{\nu i\omega}{\ell} - g\ell K_S K_T\Big(\beta\frac{S_z}{K_S^2} - \alpha\frac{T_z}{K_T^2}\Big)$$

and $X = \left(\frac{k^2 s}{\ell} - \frac{\mu^2 R}{2}\right)$

$$Y = \frac{T_z k}{2\left(i\omega + \frac{K_T}{\ell^2}\right)^2}\left(2P - \left(i\omega - \frac{K_T}{\ell^2}\right)R + \frac{\ell s}{K_T}\left(i\omega + \frac{K_T}{\ell^2}\right)\left(i\omega + \frac{K_T}{\ell^2}\right)\right)$$

$$Z = \frac{S_z k}{2\left(i\omega + \frac{K_S}{\ell^2}\right)^2}\left(2P - \left(i\omega - \frac{K_S}{\ell^2}\right)R + \frac{\ell s}{K_S}\left(i\omega + \frac{K_S}{\ell^2}\right)\left(i\omega + \frac{2K_S}{\ell^2}\right)\right)$$

The condition for (4.1) to have a solution, for A, B, and C, is that the determinant of its coefficients is zero. We make the assumptions $\nu >> K_T$ and $\nu >> K_S$, which are reasonable for most fluids. Then we find to order μ^4 that

$$P_* \left[-i\omega^3\mu^2 - \omega^2\mu^4\nu + i\omega g k^2 \alpha T_z(1-F\tau) - \frac{\nu K_S K_T}{\ell^4}\mu^2 k^2\right]$$
$$+ \frac{m^2\hat{W}^2}{2}\left[2\omega^4\mu^2 - \frac{3\nu i\omega^3}{\ell^2} + \omega^2\left(g k^2\alpha T_z(1-F\tau) - 2\mu^2 g\alpha T_z(F-\tau)\right)\right.$$
$$\left. + i\omega k^2 g\alpha T_z \frac{\nu}{\ell^2}(1-F\tau) + 2g^2\alpha^2 T_z^2 k^2 F(1-\tau)^2\right] + \frac{m^2\hat{W}^2}{2}\left[-\frac{\omega^2 k^2 s}{\ell}\right.$$
$$+ \frac{\mu^2 R\omega^2}{2} + \frac{i\omega g k^2 P}{2\left(i\omega + \frac{K_T}{\ell^2}\right)^2\left(i\omega + \frac{K_S}{\ell^2}\right)}\left(\omega^2\alpha T_z(1-F\tau) + \frac{2i\omega K_S}{\ell^2}\alpha T_z(F-1)\right.$$
$$\left. + g\alpha^2 T_z^2\frac{K_S}{\nu}(F-1)(F-\tau)\right) + \frac{g k^2 P}{2\left(i\omega + \frac{K_T}{\ell^2}\right)\left(i\omega + \frac{K_S}{\ell^2}\right)}\left(\frac{\omega^2}{\ell^2}\alpha T_z K_T(1-F\tau^2)\right.$$
$$\left. - 2i\omega\frac{K_S}{\nu}g\alpha^2 T_z^2(1-F\tau)(F-1) + \frac{K_S^2}{\nu\ell^2}g\alpha^2 T_z^2(F-1)^2\right)$$
$$- \frac{i\omega g k^2 \ell s}{2\left(i\omega + \frac{K_T}{\ell^2}\right)\left(i\omega + \frac{K_S}{\ell^2}\right)}\left(\omega^2\alpha\frac{T_z}{K_T}(F-1) - \frac{2K_S}{\nu K_T}g\alpha^2 T_z^2(F-1)^2 + \frac{2i\omega\alpha T_z}{\ell^2}(1-F\tau)\right.$$
$$\left.\left. - \frac{i\omega\alpha T_z}{\ell^2}(F-\tau)\right)\right] = 0 \qquad (4.2)$$

$$F = \frac{F_S}{F_T} \quad \text{and} \quad \tau = \frac{K_S}{K_T}.$$

Now, if we want a solution of (4.2) with $\omega = O(1)$, but not $o(1)$, then the only solution is found to be

$$\omega^2 = \frac{g k^2}{\mu^2}\left(\alpha T_z - \beta S_z\right) + g\mu^2 \qquad (4.3)$$

This gives a wave oscillating at the buoyancy frequency. It is necessary to retain the small quantity $g\mu^2$ in (4.3) in order to consistently apply the condition $\mu << \frac{1}{\ell}$. The condition for instability is that for $(g) < 0$. We then find that the system is unstable if

$$-\cos^2\theta \frac{(F_S - F_T)}{\nu(\alpha T_z - \beta S_z)}(rq + sp) > \left(q^2 + X(F-1)(1-F\tau)p^2\right) \qquad (4.4)$$

where

$$x = \frac{k^2}{\mu^2} \, \frac{\nu}{K_T} \quad , \quad \cos^2\Theta = \frac{m^2}{\mu^2}$$

and

$$p = a + 2b(1+\tau)(F-\tau)$$

$$q = 2bx(1+\tau)(F-1)(1-F\tau) - a(F-\tau)$$

$$r = 2(1+\tau)bc - a(1-F\tau) + e(F-\tau) - xd(1-F\tau) + bf + (1+\tau)h.$$

$$s = ac + 2x(1+\tau)(F-1)(1-F\tau^2)b - x(1-F\tau)(e(F-1) + d(F-\tau))$$
$$+ bh - xf(1+\tau)(1-F\tau)(F-1).$$

with

$$a = \tau^2(F-1)^2 - x(F-1)(1-F\tau)(1+\tau^2+\nu\tau) + x^2(1-F\tau)^2$$

$$b = \tau(F-1) - x(1-F\tau)$$

$$c = 2\tau(F-1)^2 + (F-\tau)(1-F\tau)$$

$$d = \tau(F-1)(F-\tau) - 2\tau(F-1)(1-F\tau) + x(1-F\tau)^2$$

$$e = \tau^2(F-1)^2 + x(1-F\tau)(1-F\tau^2) - 2x\tau(F-1)(1-F\tau)$$

$$f = 10\tau(F-1)(F-\tau) - 16\tau(F-1)(1-F\tau) - 6x(F-\tau)(1-F\tau) + 10x(1-F\tau)^2$$

$$h = 8\tau^2(F-1)^3 + 3x^2(F-1)(1-F\tau)^2 - x(1-F\tau)\left[10\tau(F-1)^2\right.$$
$$\left. + 8(1-F\tau)^2 - 10(F-\tau)(1-F\tau) + 3(F-\tau)^2\right]$$

In the stability criterion (4.4) there are two independent numbers, x and $\cos^2\Theta$, which we are free to choose. The smallest value of $\frac{F_S - F_T}{\nu(\alpha T_z - \beta S_z)}$ to give instability will obviously occur when $\cos^2\Theta \longrightarrow 1$. This implies $\frac{k^2}{\mu^2} \ll 1$. The dependence of the stability criterion on x is somewhat complicated. In fact, $(rq + sp)$ and $(q^2 + x(F-1)(1-F\tau)p^2)$ are both quintic polynomids in x. Since $\frac{k^2}{\mu^2} \ll 1$ and $\frac{\nu}{K_T} \gg 1$, $x\left(= \frac{k^2}{\mu^2}\,\frac{\nu}{K_T}\right)$ may range from zero to plus infinity.

If we let $x \longrightarrow \infty$ in (4.4) then we find it reduces to the very simple condition that there is instability if $\frac{F_S - F_T}{\nu(\alpha T_z - \beta S_z)} > \frac{1}{3}$ independent of F and τ. Now we wish to know if there are any circumstances in which we can find the system unstable for lower values of $\frac{F_S - F_z}{\nu(\alpha T_z - \beta S_z)}$

For salt fingers of heat and salt, $K_S \ll K_T$ so $\tau \ll 1$. If we make this

approximation in (4.4) we find instability if

$$\frac{F_s - F_T}{\nu\alpha T_z}\left[3x^2(F-1) + x(6F+3) + F(F-1)(3F-10)\right]$$
$$> x^2(F-1) + x(2F^2 - 2F+1) + F^2(F-1)$$

It can be shown analytically that the minimum value of the stability criterion occurs for $x \to \infty$ and the system is unstable if $\frac{F_s - F_T}{\nu\alpha T_z} > \frac{1}{3}$.

The problem of salt fingers in salt and sugar is more complicated since $\frac{\kappa_s}{\kappa_T} = O(1)$. We considered this case numerically. For salt and sugar and it has been found experimentally that $\frac{F_T}{F_s}$ 0.90 ± 0.01 (Stern and Turner 1969. We put these values in (4.4) and investigated how it behaved as a function of x. The conclusion was that the minimum value again occurred for $x \to \infty$.

Thus we find that any system of salt fingers is unstable to large scale wave perturbations if

$$\frac{F_s - F_T}{\nu(\alpha T_z - \beta S)} > \frac{1}{3} \tag{4.5}$$

Having obtained this result it is possible to trace the problem backwards in order to determine which terms affect the final result. When we do this we find that the Reynolds stress in the momentum equation (4.1a) is negligible. However, in the heat and salt equations, we find that the two forcing terms $-\hat{W}\sin\frac{x}{\ell}\frac{\partial T}{\partial z}$ and $\frac{\hat{T}}{\ell}\cos\frac{x}{\ell}\frac{\partial\psi}{\partial x}$ and equivalent terms from the salt equation are both important in determining the stability. Consequently, it is not correct to assume that the flux remains constant in the salt fingers.

There are solutions to (4.]) other than (4.3). Another solution was in fact considered, namely

$$\omega^2 = H\mu^2 \tag{4.6}$$

For this solution the real and imaginary parts of the frequency are of the same order and it was thought that this solution might prove to be more unstable than the wave perturbation which oscillates at the buoyancy frequency. This was not, however, found to be the case. The system is apparently always stable to perturbations of this frequency.

5. Discussion

Having obtained the condition (4.5) for instability, we will now compare it with some laboratory experiments. The first, and most relevant experiment we shall consider is that of Stern and Turner (1969). They use salt and sugar rather

than heat and salt as the two diffusing substances as it is experimentally simple. We shall still however refer to them as heat and salt. They make a very deep layer of fresh water with a uniform temperature gradient $T_z > 0$ and surface temperature $\bar{T}(0)$. Another very deep layer of uniform temperature $\bar{T}(0) + \Delta T$ and salinity ΔS is then placed above the first layer. The density of the upper layer is less than that of the lower, so $\alpha \Delta T > \beta \Delta S$. Salt fingers form at z - 0 as soon as the two layers are formed. The system is shown diagrammatically in Fig.1.

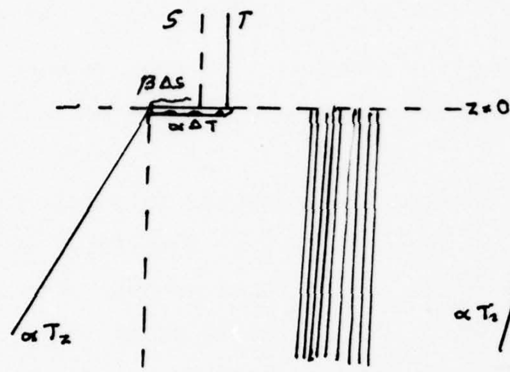

Fig.1

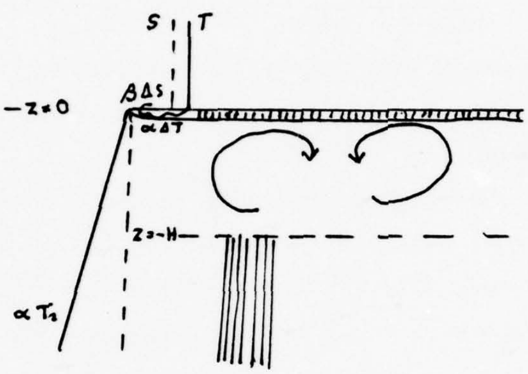

Fig.2

If the experiment is repeated, but with a smaller value of T_z, and the same values of ΔS and ΔT, then initially the salt fingers form as in Fig.1. However, after a short time the fingers between $z = 0$ and $z = -H$ become unstable and give way to a well-stirred convective layer, which is maintained by the flux through the salt finger layer at $z = 0$. This is shown in Fig.2.

If the temperature gradient T_z is reduced further the layer below $z = -H$ can become unstable. By suitable choices of the parameter ΔT, ΔS and T_z it is possible to obtain several convecting layers, each bounded above and below by a relatively thin layer of salt fingers. Layers like these have been observed in the ocean by several authors (e.g. Tait and Howe (1968, 1971) and Howe and Tait (1970)).

In order to compare this experiment with the theoretical stability criterion, it is necessary to know the salt flux, F_S, through the fingers. There is a fairly well-documented relationship between F_S and ΔS (Turner 1967)

$$F_S = C(\beta \Delta S)^{4/3}$$

The number C in the relationship may vary with $\frac{K_S}{K_T}$ and $\frac{\alpha \Delta T}{\beta \Delta S}$. For heat and salt experiments Turner (1967) found C = 10^{-1} cm/sec when $\frac{\alpha \Delta T}{\beta \Delta S} = 2$ with C decreasing only slowly as $\frac{\alpha \Delta T}{\beta \Delta S}$ increased. For the salt and sugar system Stern and Turner

(1969) found $C = 10^{-2}$ cm/sec from an experiment with $\frac{\alpha\Delta T}{\beta\Delta S} = 1.05$. Lambert and Demenkow (1972) found $C = 0.5 \rightarrow 0.75\ 10^{-3}$ cm/sec from an experiment with $\frac{\alpha\Delta T}{\beta\Delta S} = 1.25$. Thus for salt and sugar the salt flux changes quite considerably $\frac{\alpha\Delta T}{\beta\Delta S}$ and the salt flux is reduced for larger values of $\frac{\alpha\Delta T}{\beta\Delta S}$. Applying the relationship (5.1) to Stern and Turner's (1969) experiment, using their value of $C = 10^{-2}$ cm/sec, we find that if $\frac{F_S - F_T}{\gamma(\alpha T_z - \beta S_z)} \simeq 2.8$ the system is unstable and that if $\frac{F_S - F_T}{\gamma(\alpha T_z - \beta S_z)} \simeq 1.2$ the system is stable. Considering the uncertainties in the experiments the value for instability of about 2 is close enough to the theoretical 1/3 for the instability which is observed to be the collective instability.

The second experiment we shall consider isolates the thin salt finger layer that exists between two convecting regions. A very deep layer of uniform tempearture $T_0 + \frac{\Delta T}{2}$ and salinity $S_0 - \frac{\Delta S}{2}$ is placed above another deep layer of temperature $T_0 + \frac{\Delta T}{2}$ and salinity $S_0 - \frac{\Delta S}{2}$. The system is shown in Fig.3.

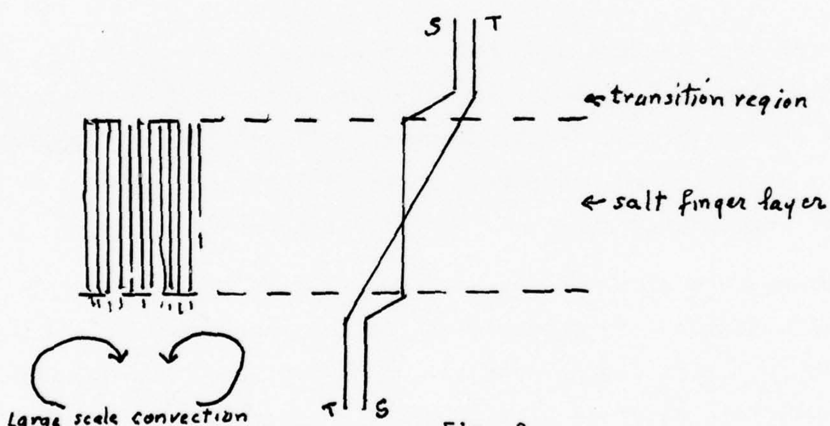

Fig. 3

Linden (1973) performed this experiment using heat and sugar and he found values of $\frac{F_S - F_T}{\gamma(\alpha T_z - \beta S_z)}$ ranging from $0.2 \rightarrow 1.9$ in the salt finger layer. The fingers are observed to be stable thus the theory of § 4 predicts that $\frac{F_S - F_T}{\gamma(\alpha T_z - \beta S_z)}$ should be less than 1/3. The fact that the value actually obtained is close to 1/3 suggests that the region may actually be marginally stable. Lambert and Demenkow (1972) performed this experiment with salt and sugar. They found, using a mixture of theory and experiment, that $\frac{F_S - F_T}{\gamma(\alpha T_z - \beta S_z)} \simeq 2 \times 10^{-3}$ for their experiments. This suggests that their salt finger region is very stable. The

reason it is so stable is probably that they did their experiments at comparatively large values of $\frac{\alpha \Delta T}{\beta \Delta S}$, which reduced the salt flux through the fingers, so making them more stable.

More recently some experiments have been done by Linden (1978) in which he set up a region of linear salt and sugar gradients of a fixed thickness. He let this region develop in time and he found that, in some circumstances, the region became unstable. The region developed convection regions separated by thin salt finger layers. There is some difficulty in applying the theory to this experiment as the temperature and salinity gradients are changing as the experiment proceeds. However, if we take initial values, we do find that $\frac{F_S - F_T}{\nu(\alpha T_z - \beta S_z)} \gtrsim 1/3$ is an approximate measure of whether or not the system will be unstable.

So we see that the results of the experiments are not in disagreement with the stability theory. We have shown here that the two-dimensional salt fingers will be unstable if $\frac{F_S - F_T}{\nu(\alpha T_z - \beta S_z)} > 1/3$. The main result of this study is that it has shown that the collective instability can be put on a firm footing, and does not have to rest on shaky physical assumptions. Before this instability is fully understood it will be necessary to look at the energetics of the process, in order to see where the energy that drives the large scale motion is coming from and whether this instability increases or decreases the heat and salt fluxes through the system.

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References

Howe, M.R. and R.I.Tait 1968 Deep Sea Res. 17: 963-972.

Huppert, H.E. and P.C.Manins 1973 Linking conditions for salt-fingering at an interface. Deep Sea Res. 2: 315-323.

Lambert, R.B. and J.W.Demenkow 1972 On the vertical transport due to salt fingers. J.F.M. 54: 627-640.

Linden, P.F. 1973 On the structure of salt fingers. Deep Sea Res. 20: 325-340.

Linden, P.F. 1978 The formation of banded salt finger structure. J.Geo.Res. 83: 2902-2912.

Stern, M.E. 1969 Collective instability of salt fingers. J.F.M. 35: 209-218.

Stern, M.C. 1975 Ocean Circulation Physics, ch.XI Academic, New York.

Stern, M.E. and J.S.Turner 1969 Salt fingers and convecting layers. Deep Sea Res. 16: 497-511.

Stommel, H., A.Arons and D.Blanchard 1956 An oceanographic curiosity: the perpetual salt fountain. Deep Sea Res. 3: 152-153.

Tait, R.I. and M.R.Howe 1968 Deep Sea Res. 15: 275-280.

Tait, R.I. and M.R.Howe 1971 Nature, 231: 178-179.

Turner, J.S. 1967 Salt fingers across a density interface. Deep Sea Res. 14: 599-612.

Williams, A.J. 1974 Salt fingers observed in the Mediterranean outflow. Science 185: 941-943.

MEAN FIELD EQUATIONS FOR CERTAIN MOMENTS OF THE MAGNETIC FIELD

Glenn R. Ierley

Introduction

A great deal of work in dynamo theory has centered on the kinematic problem of identifying classes of fluid motion that may generate magnetic fields and it is now well-known that helical homogeneous isotropic turbulence can do so. Kraichnan and Knobloch have separately shown that for a turbulent velocity field with large scale persistent helicity fluctuations but zero net helicity, the mean magnetic field may be negatively diffused to small spatial scales. Equilibration of the magnetic field requires solution of the fully interacting dynamical problem and not surprisingly, much less can be said about this. In the first part of this paper we find the kinematic mean field equation for the Lorentz force with the motivation that in the weak field regime the results may suggest an iterative scheme for finite amplitude equilibration although it is important to bear in mind as Kraichnan has noted that it may be the fluctuations of the Lorentz force that are important or as Knobloch has mentioned, that the magnetic helicity may play a significant role in suppressing helicity in the velocity field.

In the second part of the paper we consider the Lagrangian evolution equation for the magnetic energy and conclude that kinematic considerations alone result in magnetic energy being concentrated in large scales even in

the instance that the mean magnetic field is negatively diffused.

Lorentz Force

For a stochastic differential equation of the form (suppressing spatial dependence)

(1) $$\left[\frac{\partial}{\partial t} + L'(t) + \bar{L}(t)\right]\left(\bar{f}(t) + f'(t)\right) = 0$$

where bar denotes mean, and prime denotes zero mean fluctuations; the exact solution for the mean field f(t) may be written (Knobloch 1977), assuming f'(0) = 0, as

(2) $$\left[\frac{\partial}{\partial t} + \bar{L}(t)\right]\bar{f}(t) = \int_0^t dt' \langle L'(t) \exp_0\left\{-\int_{t'}^{t} dt_1\, u_0(t, t_1)\right.$$
$$\left.(1-A)\, L'(t_1)\right\} u_0(t_1, t')\, L'(t') \rangle\, \bar{f}(t').$$

U_0 satisfies the equation $\left[\frac{\partial}{\partial t} + \tau(t)\right] u_0(t,0) = 0$ with $U_0(0,0) = 1$. The subscript "0" on the exponential denotes a time ordered product (latest time to the left) in the expansion, and A is an operator which takes the average of everything to the right. In the limit of short autocorrelation time i.e.

$$\langle L'(t)\, L'(t') \rangle \propto \langle L'(t)\, L'(t) \rangle\, \delta(t-t')$$

we have

(3) $$\frac{\partial}{\partial t}\bar{f}(t) = \langle L'(t)\, L'(t) \rangle\, \tau\, \bar{f}(t).$$

The result may also be obtained from first order smoothing.

We do not include any sure operator $\tau(t)$; thus molecular diffusivity is ignored. Moffatt has questioned the convergence of the coefficients in the expansion of Eq.(2) in this circumstance and the point is not fully resolved. For the Lorentz force the operator L' is a fourth rank tensor obtained from the induction equation.

(4) $$\frac{\partial}{\partial t} B_j = -\partial_i\left(V_i B_j - B_i V_j\right)$$

where $B_j = B_j(x,t)$. Writing the same equation for $B_K' = B_K(x',t)$, multiplying the first by B_K', the second by B_j we find

(5) $$\frac{\partial}{\partial t}\left(B_j B_{K'}\right) = L_{jKmn}\left(B_m B_{n'}\right)$$

where $L_{jKmn} = -\delta_{jm}\delta_{Kn}\left(V_i \partial_i + V_{\ell'}\partial_{\ell'}\right) + \delta_{im}\delta_{Kn}\left(\delta_i V_j\right) + \delta_{jm}\delta_{\ell n}\left(\partial_{\ell'} V_{K'}\right).$

Thus Eq.(3) becomes

(6) $$\frac{\partial}{\partial t}\langle B_j B_{K'} \rangle = \langle L_{jKmn}(t)\, L_{mnpq}(t) \rangle\, \tau \langle B_p B_q' \rangle.$$

The result is

$$\frac{\partial}{\partial t}\langle B_j B_{k'}\rangle = \eta(\partial_i\partial_i + \partial_{i'}\partial_{i'})\langle B_j B_{k'}\rangle + 2\eta_{im}(\partial_i\partial_{m'}\langle B_j B_{k'}\rangle)$$

$$(7)\quad -\alpha_2\varepsilon_{lmj}\partial_i\langle B_m B_k'\rangle - \alpha_2\varepsilon_{imk}\partial_i\langle B_j B_{m'}\rangle - (\partial_{m'}\eta_{ik})\partial_l\langle B_j B_{m'}\rangle$$

$$-(\partial_m\eta_{ji})\partial_i\langle B_m B_k'\rangle - (\partial_i\eta_{jm})\partial_{m'}\langle B_i B_{k'}\rangle + 2(\partial_l\partial_{m'}\eta_{jk})\langle B_i B_{m'}\rangle - (\partial_i\eta_{mk})\langle B_{j,m} B_i\rangle$$

where

$$(8)\quad \int_0^\infty v_j(\vec{x},t)\,v_m(\vec{x},t')\,dt' = \eta_{jm}(|\vec{x}-\vec{x}'|) \sim \frac{u^2}{2\lambda^2} r_j r_m + u^2\left(1-\frac{r^2}{\lambda^2}\right)\delta_{jm} + \frac{1}{2}(\alpha_2 + r^2\alpha_3)\varepsilon_{jmk} r_k.$$

and $r = |\vec{x} - \vec{x}'|$, λ is the Taylor microscale.
u^2 is the mean square velocity.

Now we operate with ∂_l' on both sides obtaining: (dropping primes on B and its derivatives)

$$(9)\quad \left[\frac{\partial}{\partial t} + \eta(\nabla^2 + \nabla'^2)\right]\langle B_j B_{k,l}\rangle = -\frac{1}{2}\alpha_2\varepsilon_{iml}\left[\langle B_{j,i} B_{k,m}\rangle - \langle B_{j,m} B_{k,i}\rangle\right] +$$

$$2u^2\langle B_{j,i} B_{k,il}\rangle - \frac{1}{2}\alpha_2\varepsilon_{jmi}\langle B_i B_{k,lm}\rangle + \frac{u^2}{\lambda^2}\left[\frac{1}{2}(\delta_{lm}\delta_{ij} + \delta_{lj}\delta_{im}) - 2\delta_{il}\delta_{jm}\right]\langle B_i B_{k,m}\rangle$$

$$+\frac{1}{2}\alpha_2\varepsilon_{mki}\langle B_{i,l} B'_{j,m}\rangle - \frac{u^2}{\lambda^2}\left[\frac{1}{2}(\delta_{lk}\delta_{im} + \delta_{lm}\delta_{ik}) - 2\delta_{il}\delta_{mk}\right]\langle B_{j,m} B_i\rangle$$

$$-\alpha_2\varepsilon_{imj}\langle B_{m,i} B_{k,l}\rangle - \alpha_2\varepsilon_{imk}\langle B_j B_{m,il}\rangle + \frac{1}{2}\alpha_2\varepsilon_{ikm}\langle B_{j,i} B_{m,l}\rangle$$

$$-\frac{1}{2}\alpha_2\varepsilon_{jim}\langle B_m B_{k,il}\rangle + 2(\partial_{l'}\partial_{m'}\partial_i\eta_{jk})\langle B_i B_m\rangle$$

$$-\frac{u^2}{\lambda^2}\left[\frac{1}{2}(\delta_{li}\delta_{mk} + \delta_{lk}\delta_{mi}) - 2(\delta_{lm}\delta_{ik})\right]\langle B_{j,i} B_m\rangle$$

$$+\frac{u^2}{\lambda^2}\left[\frac{1}{2}(\delta_{lj}\delta_{mi} + \delta_{li}\delta_{mj}) - 2\delta_{lm}\delta_{ij}\right]\langle B_m B_{k,i}\rangle$$

$$-2\frac{u^2}{\lambda^2}\left[\frac{1}{2}(\delta_{ij}\delta_{mk} + \delta_{ik}\delta_{mj}) - 2\delta_{im}\delta_{jk}\right]\langle B_i B_{m,l}\rangle.$$

Now things are simplified by contracting both sides with $\varepsilon_{kml}\varepsilon_{pjm}$ to form $\langle B\times(\nabla\times B)\rangle$ on the left-hand side. It may be noted the terms in u^2/λ^2 give rise to a contribution

$$-2\frac{u^2}{\lambda^2}\left[\langle B\times(\nabla\times B)\rangle\right] + 2\frac{u^2}{\lambda^2}\nabla\langle B^2\rangle.$$

The α_2 terms give $\alpha_2\nabla\times\langle B\times(\nabla\times B)\rangle$, and the term $2(\partial_{l'}\partial_{m'}\partial_i\eta_{jk})$ vanishes upon contraction of the indices. The contribution $\nabla\langle B^2\rangle$ may be eliminated using the operator $\left[\frac{\partial}{\partial t} - (\lambda_2 + \eta\nabla^2)\right]$ which annihilates $\langle B^2\rangle$ (, a result previously found by Knobloch (Knobloch 1978 a). The final answer may be written

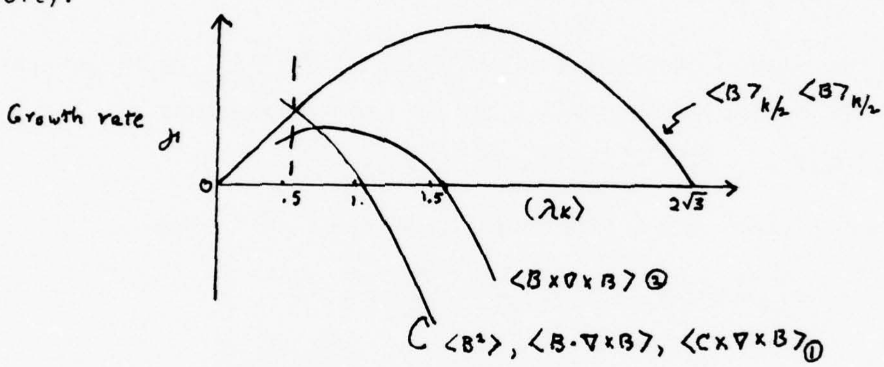
$$\left[\frac{\partial}{\partial t} - \left(\frac{2}{3}\tau\langle\omega^2\rangle + \frac{1}{3}\tau\langle u^2\rangle\nabla^2\right)\right]\left[\frac{\partial}{\partial t} - \frac{1}{3}\tau\langle u^2\rangle\nabla^2 + \right.$$

$$\left. \frac{2}{15}\tau\langle\omega^2\rangle - \frac{1}{3}\tau\langle h\rangle\nabla\times\right]\langle B\times(\nabla\times B)\rangle = 0. \tag{10}$$

We note there are two modes for the Lorentz force where the second is strikingly like the α-effect generation of the mean field $\langle B\rangle$. At this order the following equations are also satisfied:

$$\left[\frac{\partial}{\partial t} - \left(\frac{2}{3}\tau\langle\omega^2\rangle + \frac{1}{3}\tau\langle\omega^2\rangle\nabla^2\right]\left\{\begin{matrix}\langle B^2\rangle \\ \langle B\cdot\nabla\times B\rangle\end{matrix}\right\} = 0 \tag{11}$$

It is instructive to plot the growth rates of the above moments at wave vector k and the mean magnetic field at k/2 as a function of k for a state of maximum helicity.



The following features should be noted: for $(\lambda K) < .504$ the variance of the mean field grows faster than the field itself. Thus it no longer makes sense to identify the real field with the mean field. For $\lambda K > 2\sqrt{3}$ the mean field decays which is also uninteresting. Finally only for $.504 < \lambda K < 1.608$ does the Lorentz force grow, for $1.608 < \lambda K < 2\sqrt{3}$ kinematic results indicate decay of the Lorentz force (as well as the magnetic helicity for $(\lambda K) > 1$. For a general spectral distribution of $\langle\vec{B}\rangle$ in k space the picture is unclear but it seems quite plausible that there is still a range of λK in which the growth of $\langle\vec{B}\rangle$ can only be limited by fluctuations of the Lorentz force at least initially.

Interestingly, if one calculates $\langle\vec{B}\rangle \times (\nabla\times\langle\vec{B}\rangle)$ the result is:

$$i\, exp\left(-\frac{1}{3}\tau\langle u^2\rangle\frac{K^2}{2}t\right)B^2(\vec{K},0)\,\hat{k}\,.$$

This is obtained from the exact solution of the Rourier transformed dynamo equation with the condition that $\vec{B}'(\vec{K},0) = 0$. In this case a growing Lorents force arises solely from $\langle\vec{B}'\times(\nabla\times\vec{B}')\rangle$, the nonzero average of the product of fluctuating fields.

Mean Square Magnetic Field

For the induction equation

(12) $\partial_t B_i(\vec{x},t) = -\partial_j \left(V_j(\vec{x},t)\, B_i(\vec{x},t) - B_j(\vec{x},t) V_i(\vec{x},t) \right)$

in an incompressible fluid one may write the exact Cauchy solution

(13) $$B_i(\vec{x},t) = \frac{\partial x_i(\vec{a},t)}{\partial a_j} B_j(\vec{a},0)$$

where

$$x_i(\vec{a},0) = a_i \quad \text{and} \quad \frac{\partial x_i(\vec{a},t)}{\partial a_j} \equiv \delta_{ij} + \frac{\partial \xi_i(\vec{a},t)}{\partial a_j}$$

$\xi_i(\vec{a},t) = x_i(\vec{a},t) - a_i$, the relative displacement of a fluid particle starting at $\vec{a}$ at t = 0.

For the convection of the quantity $B_i(\vec{x},t)\, B_j(\vec{x},t)$ it is easily shown that the solution is simply the product of the Cauchy solutions

(14) $$B_i(\vec{x},t) B_j(\vec{x},t) = \frac{\partial x_i}{\partial a_m} \frac{\partial x_j}{\partial a_n} B_m(\vec{a},0)\, B_n(\vec{a},0).$$

If we expand each factor on the right-hand side about $\vec{x}$, then

(15) $$\frac{\partial x_i}{\partial a_m} B_m(\vec{a},0) \sim \frac{\partial x_i}{\partial a_m} B_m(\vec{x},0) - \xi_K \frac{\partial x_i}{\partial a_m} \frac{\partial B_m}{\partial x_K}(\vec{x},0) + \frac{1}{2} \xi_K \xi_{K'} \frac{\partial x_i}{\partial a_m} \frac{\partial^2 B_m(\vec{x},0)}{\partial x_K \partial x_{K'}}$$

Thus

(16) $$B_i(\vec{x},t) B_j(\vec{x},t) \sim \frac{\partial x_i}{\partial a_m} \frac{\partial x_j}{\partial a_n} B_m(\vec{x},0) B_n(\vec{x},0) - \xi_K \frac{\partial x_i}{\partial a_m} \frac{\partial x_j}{\partial a_n} \frac{\partial}{\partial x_K} (B_m(\vec{x},0) B_n(\vec{x},0)) + \tfrac{1}{2} \xi_K \xi_{K'} \frac{\partial x_i}{\partial a_m} \frac{\partial x_j}{\partial a_n} \Big[B_m(\vec{x},0) \frac{\partial^2 B_n(\vec{x},0)}{\partial x_K \partial x_{K'}} + B_n(\vec{x},0) \frac{\partial^2 B_m(\vec{x},0)}{\partial x_K \partial x_{K'}} + \frac{\partial B_m}{\partial x_K}(\vec{x},0) \frac{\partial B_n}{\partial x_{K'}}(\vec{x},0)$$

Now we average both sides assuming statistical independence of the field at t = 0 from all realizations of the velocity field for $t \geq 0$. The assumption of homogeneity and isotropy allows us to write, for example,

$$\left\langle \frac{\partial x_i}{\partial a_m} \frac{\partial x_j}{\partial a_n} \right\rangle = A\, \delta_{im} \delta_{jn} + B\, \delta_{ij} \delta_{mn} + C\, \delta_{in} \delta_{jm},$$

and after contraction of indices i, j and n, m, we find

$$(A + 3B + C) = \tfrac{1}{3} \left\langle \left(\frac{\partial x_i}{\partial a_m} \right)^2 \right\rangle .$$

In this fashion we evaluate all the terms involving moments of the velocity field. Finally Eq.(16) is contracted with a factor of δ_{ij} . The result may be written as

$$(17)\quad \langle B^2(\vec{x},t)\rangle = \tfrac{1}{3}\left\langle\left(\frac{\partial x_i}{\partial a_j}\right)^2\right\rangle\langle B^2(\vec{x},0)\rangle + \tfrac{1}{15}\left[2\left\langle(\xi_k)^2\left(\frac{\partial x_i}{\partial a_j}\right)^2\right\rangle - \left\langle\xi_k\frac{\partial x_i}{\partial a_k}\,\xi_j\frac{\partial x_i}{\partial a_j}\right\rangle\right]\nabla_x^2\langle B^2(\vec{x},0)\rangle + \tfrac{1}{10}\left[3\left\langle\xi_k\frac{\partial x_i}{\partial a_k}\,\xi_j\frac{\partial x_i}{\partial a_j}\right\rangle - \left\langle(\xi_k)^2\left(\frac{\partial x_i}{\partial a_j}\right)^2\right\rangle\right]\partial_\ell\partial_m\langle B_\ell(\vec{x},0)\,B_m(\vec{x},0)\rangle$$

To estimate the time dependence of the coefficients we assume (c.f. Kraichnan 1976b)

$$(17a)\quad \vec{\xi} = \sum_{s=1}^{N}\vec{\xi}_s \quad \text{and} \quad \frac{\partial x_1}{\partial a_1} = \prod_{s=1}^{N} W_s$$

where $\langle W_s\rangle = 1$ and $\langle W_s^2\rangle > 1$. That is, the displacement is the sum of small displacements from a large number, N, of independent eddies while the diagonal component of the strain is the product of N independent strains again from independent eddies. Using the fact that

$$\frac{\partial x_i}{\partial a_j} = \delta_{ij} + \frac{\partial \xi_i}{\partial a_j}$$

we see that

$$\left\langle\frac{\partial x_i}{\partial a_j}\right\rangle = \delta_{ij} + \frac{\partial}{\partial a_j}\langle\xi_i\rangle.$$

The second term however must vanish by homogeneity and diagonal components of the strain must therefore average to one. The model above reproduces this result as

$$\left\langle\frac{\partial x_1}{\partial a_1}\right\rangle = \left\langle\prod_{s=1}^{N} W_s\right\rangle = \langle W_s\rangle = 1$$

and the fact that the W's are fluctuating quantities implies $\langle W_s^2\rangle$. The same reasoning implies off diagonal components might be represented as where $\langle U_s\rangle = 0$ ensuring $\left\langle\frac{\partial x_1}{\partial a_2}\right\rangle = 0$.

It is by no means clear that $\langle U_s^2\rangle$ is less than one, however, we assert that it is plausible that

$$\langle W_s^2\rangle > \langle U_s^2\rangle$$

which condition will allow us to neglect off-diagonal contributions in (17). From the assumption that the displacement in each eddy is independent of the strain a coefficient of the form

$$\left\langle(\xi_k)^2\left(\frac{\partial x_i}{\partial a_j}\right)^2\right\rangle$$

may be written as $\quad 9\left\langle(\xi_2)^2\left(\frac{\partial x_1}{\partial a_1}\right)^2\right\rangle + 18\left\langle(\xi_2)^2\left(\frac{\partial x_1}{\partial a_3}\right)^2\right\rangle.$

The term $\left\langle\xi_k\frac{\partial x_i}{\partial a_k}\,\xi_j\frac{\partial x_i}{\partial a_j}\right\rangle$ must by isotropy have k = j and thus becomes $3\left\langle(\xi_2)^2\left(\frac{\partial x_i}{\partial a_1}\right)^2\right\rangle + 6\left\langle(\xi_2)^2\left(\frac{\partial x_1}{\partial a_3}\right)^2\right\rangle$ where we have used the equivalence of

$\langle(\xi_2)^2(\frac{\partial x_1}{\partial a_2})^2\rangle$ with $\langle(\xi_2)^2(\frac{\partial x_1}{\partial a_2})^2\rangle$. It is now apparent that the last two terms in (17) may be written as

$$\left[\langle(\xi_2)^2(\frac{\partial x_1}{\partial a_1})^2\rangle + 2\langle(\xi_2)^2(\frac{\partial x_1}{\partial a_3})^2\rangle\right]\nabla_x^2\langle B^2(\vec{x},0)\rangle + [0]\,\partial_\ell\,\partial_m\langle B_\ell B_m(\vec{x},0)\rangle.$$

Observe that off-diagonal terms make a positive definite contribution to the diffusive term (which as noted above we will neglect), while the coefficient of the mixed derivative term vanishes given the independence of displacement and strain. In the instance of negative diffusivity of the mean magnetic field this is invalid but we appeal then to the assertion that $\langle\omega_s^2\rangle > \langle U_s^2\rangle$ for the neglect of the last term.

From (17a) we estimate that

$$\frac{1}{3}\langle\left(\frac{\partial x_i}{\partial a_j}\right)^2\rangle \sim \langle\left(\frac{\partial x_1}{\partial a_1}\right)^2\rangle = \langle\omega_s^2\rangle^N.$$

We claim $N \propto \frac{U_0}{\ell_0}t$ where U_0 and ℓ_0 are characteristic velocity and length scale of the eddies thus

$$\frac{1}{3}\langle\left(\frac{\partial x_i}{\partial a_j}\right)^2\rangle \propto e^{\alpha t}$$

where α is positive definite since in $\langle\omega_s^2\rangle > 0$. Similarly a term

$$\langle(\xi_2)^2\frac{\partial x_1}{\partial a_1}\rangle \text{ is proportional to } N\langle(\xi_{2s})^2\omega_s\rangle$$

reflecting linear growth in time. Terms proportional to N^2 here vanish as they involve averages like

$$\langle\xi_{2s}\omega_s\rangle\langle\xi_t\rangle.$$

We need $\langle(\xi_2)^2(\frac{\partial x_1}{\partial a_1})^2$ which is easily seen to be

$$N\langle(\xi_{2s})^2\omega_s^2\rangle\langle\omega_t^2\rangle^{N-1}.$$

Collecting our results we obtain:

$$\text{(18)}\quad \langle B^2(\vec{x},t)\rangle \sim e^{\alpha t}\langle B^2(\vec{x},0)\rangle + \beta t e^{\alpha t}\nabla^2\langle B^2(\vec{x},0)\rangle.$$

In the case of a uniform field, B_0, the second term vanishes and we see asymptotically

$$\ln\left[\frac{\langle B^2(\vec{x},t)\rangle}{B_0^2}\right] = \alpha t$$

which is confirmed in Kraichnan's numerical experiments for several prescriptions of the turbulent field. Differentiating the equation above we find

$$\partial_t\langle B^2(\vec{x},t)\rangle \sim \alpha e^{\alpha t}\langle B^2(\vec{x},0)\rangle + (\beta + \alpha\beta t)e^{\alpha t}\nabla^2\langle B^2(\vec{x},0)\rangle$$

and inverting (18) to find $\langle B^2(\vec{x},0)\rangle$ in terms of $\langle B^2(\vec{x},t)\rangle$ we obtain

$$\text{(19)}\quad \frac{\partial}{\partial t}\langle B^2(\vec{x},t)\rangle = \alpha\langle B^2(\vec{x},t)\rangle + \beta\nabla^2\langle B^2(\vec{x},t)\rangle$$

where β is a positive definite constant $(\langle(\xi_{2s})^2 W_s^2\rangle)$. In the presence of strong persistent large scale helicity fluctuations Kraichnan argues (Kraidhnan, GFD 1978)

$$\langle(\xi_2)^2 \frac{\partial x_1}{\partial a_1}\rangle \propto -t^2$$

since even though sequential realizations of the velocity field are completely uncorrelated, the negative correlation of large values of ξ^2 with large values of the strain makes probable an even more negative correlation in the next interval. In a fluid with zero net helicity then $\eta(t) \propto -t$ and negative diffusion obtains. In these same circumstances we conjecture

$$\langle(\xi_2)^2\left(\frac{\partial x_1}{\partial a_1}\right)^2\rangle \propto \beta t^2$$

While the matter of the exponent is uncertain, β is surely positive definite thus positive diffusion of the mean square magnetic field obtains <u>even</u> in the instance of negative diffusion of the mean field and possibly the diffusion of $\langle B^2\rangle$ is greatly enhanced.

The conclusion is that from kinematic considerations alone the magnetic energy is always in large scales prompting us to note that the contribution Of $\langle B'B'\rangle$ to $\langle B^2\rangle$ completely dominates that of $\langle B\rangle\langle B\rangle$ when the latter is negatively diffused.

Finally we note that in reproducing the mean field approximation to the mean square field in terms of the Eulerian velocity Keller has shown that there is a difficulty of a rather general nature connected with the fact that an equation like (17) is not valid for $t \to \infty$, in particular we have

$$\langle B^2(\vec{x},t)\rangle = (1+\tfrac{1}{3}\tau\langle w^2\rangle t)\langle B^2(\vec{x},0)\rangle + \tfrac{t}{3}\tau\langle u^2\rangle\nabla^2\langle B^2(\vec{x},0)\rangle,$$

and differentiating we see

$$\partial_t\langle B^2(\vec{x},t)\rangle = \tfrac{1}{3}\tau\langle w^2\rangle\langle B^2(\vec{x},0)\rangle + \tfrac{1}{3}\tau\langle u^2\rangle\nabla^2\langle B^2(\vec{x},0)\rangle.$$

To reproduce the first order Eulerian result for all t we need to obtain $\langle B^2(\vec{x},t)\rangle \sim \langle B^2(x,0)\rangle$ from the inversion. This does not obtain for $t \to \infty$.

<u>CONCLUSIONS</u>

For further exploration we feel it would be useful to examine

$$\frac{\langle|\vec{B}\times(\nabla\times\vec{B})|^2\rangle}{|\langle\vec{B}\times(\nabla\times\vec{B})\rangle|^2}$$

to see under what circumstances the mean Lorentz force dominates the fluctuating Lorentz force although it would perhaps be most useful to do this in a Lagrangian framework as was done for $\langle B^2\rangle$. As to the applicability of (10)

we expect for realistic turbulence the coefficients may be radically changed by renormalization but there is at present no means to explicitly calculate appropriate values even given the exact form of higher correction terms. Thus in a pragmatic sense we feel that the development of the formalism for large molecular diffusivity, while not completely straightforward, might be more useful for quantitative applications in appropriate physical systems.

The sensitivity of the results to departures of the turbulent field from homogeneity or isotropy is not clear although the length scale of inhomogeneities is an important consideration. This problem is naturally of some interest in regard to dynamical interactions and warrants further investigation.

Acknowledgments

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References

Batchelor, G.K. 1960 The Theory of Homogeneous Turbulence. Cambridge University Press.

Frisch, U. 1968 Probabilistic Methods in Applied Mathematics (ed.A.T.Bharucha-Reid). New York: Academic Press.

Keller, J. (Private communication).

Knobloch, E. 1977 J.Fluid Mech. 83: 129.

Knobloch, E. 1978a Astrophys.J. 220: 330.

Knobloch, E. 1978b Astrophys.J. (to be published 1 Nov.).

Knobloch, E. "Nonlinear Theory of Turbulent Diffusion of Magnetic Fields" (in preparation).

Knobloch, E. 1978 Thesis Harvard University.

Kraichnan, R.H. 1974 J.Fluid Mech. 64: 47.

Kraichnan, R.H. 1976a J.Fluid Mech. 75: 657.

Kraichnan, R.H. 1976b J.Fluid Mech. 77: 753.

Kraichnan, R.H. GFD Lecture 1978 (author's notes) c.f. this volume for abstract.

Moffatt, H.K. 1974 J.Fluid Mech. 65: 1.

Moffatt, H.K. 1978 "Magnetic Field Generation in Electrically Conducting Fluids", pp.156, 169-187. Cambridge: Cambridge University Press.

ON TURBULENT EKMAN LAYERS - THE EFFECT OF FINITE ROSSBY NUMBER ETC.

Shigeki Mitsumoto

1. Introduction

The Ekman layer is the lowest part of the troposphere, or the upper part of the ocean, in which the flow is under the simultaneous influence of pressure, Coriolis, and frictional forces, which are equally dominant.

For the theoretical study of the turbulent Ekman layer flow, or in general, of the boundary layer flow, the greatest problem is the turbulent mixing process by small scale motion. This is so complicated that meteorologists or oceanographers in their search for a suitable scheme have chosen to parametize it with some mean field values, relating the Reynolds stress with the shear of the mean velocity by an "eddy diffusivity" Km as $\tau = K_m \frac{\partial u}{\partial z}$. A variety of parametization schemes have been suggested, which resulted in a variety of theories for the Ekman layer.

For example, in his study of atmospheric turbulence, Ellison (1956) assumed that Km is proportional to height and obtained the vertical wind profile in the Ekman layer.

However, the assumption of eddy diffusivity itself has been questionable; according to some observational data, it is sometimes found to be negative or even infinite. Some alternate methods, with not too much sophistication to apply to the actual situation, but based on more physically reasonable considerations, has been sought.

Malkus (1978) presented quite a unique method to obtain the velocity profile of the turbulent, neutrally stratified, one-dimensional steady channel flow, which is based on two simple assumptions; (i) the mean velocity profile

is free from any inflexion points and (ii) spectral smoothness of the Reynolds stress. The remarkable point in this theory is that the prodile of the mean velocity and the Reynolds stress are obtained uniquely without any assumption of eddy diffusivity.

The object of this study is to apply Malkus' theory to the two-dimensional turbulent Ekman layer.

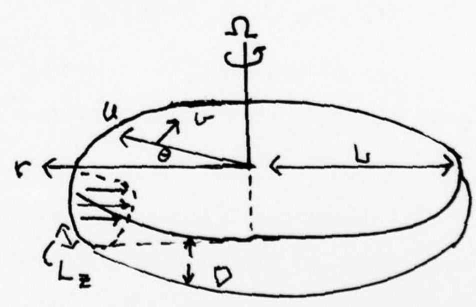
Laboratory experiments for turbulent Ekman layers were carried out by Caldwel *et al.* and by Kreider (1973). Kreider compares the velocity profile with some theoretical curves and those obtained in field observation. Among them, the curve derived from Ellison's theory which is simpler than others, fits Kreider's data best. However, Kreider's curve shows dependence on R_{oa}, the apparatus Rossby number, representing the effect of centrifugal force, which is not considered in Ellison's theory. The experimental curve by Caldwel *et al.* does not show the systematic dependence on R_{oa}.

Thus, another purpose of this study is to investigate the effect of finite Rossby number on the velocity profile and to give a quantitative explanation to Kreider's results. We do this by extending Ellison's theory to include Rossby number, since Ellison's curve is most consistent, at least qualitatively, with Kreider's data.

II. Basic Equation for the Turbulent Ekman Layer and its Nondimensionalization

The Momentum Equation including Reynolds stress in (r, θ, z) coordinate rotating with angular velocity Ω (= const) is written as follows if we assume;

1. Mean Field is axisymmetric



2. $\bar{w} \simeq 0$ (quasi-two-dimensional)

Fig.1

$$\underline{\bar{u}\frac{\partial \bar{u}}{\partial r}} - \underline{\frac{1}{r}\bar{v}^2} - \underline{2\Omega\bar{v}} + \underline{\frac{1}{\rho}\frac{\partial}{\partial r}}\bar{p}* - \nu\underline{\frac{\partial^2 \bar{u}}{\partial z^2}}$$
$$+ \left\{ \underline{\frac{\partial}{\partial r}(\overline{u'^2})} + \underline{\frac{1}{r}\frac{\partial}{\partial \theta}(\overline{u'v'})} + \underline{\frac{\partial}{\partial z}(\overline{u'w'})} - \underline{\frac{u'^2+v'^2}{r}} \right\} = 0 \qquad (2.1)$$

$$\underline{\bar{u}}\,\frac{\partial \underline{\bar{v}}}{\partial \underline{r}} + \frac{1}{\underline{r}}\underline{\bar{u}}\underline{\bar{v}} + 2\Omega\underline{\bar{u}} - \nu\left(\frac{\partial^2 \underline{\bar{v}}}{\partial \underline{r}^2} + \frac{1}{\underline{r}}\frac{\partial \underline{\bar{v}}}{\partial \underline{r}} - \frac{\underline{\bar{v}}}{\underline{r}^2} + \frac{\partial^2 \underline{\bar{v}}}{\partial \underline{z}^2}\right)$$
$$+\left\{\frac{\partial}{\partial \underline{r}}(\overline{\underline{u}'\underline{v}'}) + \frac{1}{\underline{r}}\cdot\frac{\partial}{\partial\theta}(\overline{\underline{v}'^2}) + \frac{\partial}{\partial \underline{z}}(\overline{\underline{v}'\underline{w}'})\right\} = 0 \qquad (2.2)$$

where

$$(\underline{u},\underline{v}) = (\underline{\bar{u}},\underline{\bar{v}}) + (\underline{u}',\underline{v}'),$$

under bar denotes dimensional values,

and $\bar{\underline{p}}^* = \bar{\underline{p}} - \frac{1\Omega}{2}r^2$)reduced pressure).

Scaling

It is natural to scale $\underline{r}$, $\left(\frac{\bar{\underline{u}}}{\bar{\underline{v}}}\right)$, $\left(\frac{\underline{u}'}{\underline{v}'}\right)$ and $\bar{\underline{p}}^*$ in the following way:

$\underline{r} \equiv L r$ (L: radius of the rotating tank)

$\left(\frac{\bar{\underline{u}}}{\bar{\underline{v}}}\right) \equiv G\left(\frac{\bar{u}}{\bar{v}}\right)$ (G: speed of flow relative to rotating tank)

$\left(\frac{\underline{u}'}{\underline{v}'}\right) \equiv v^*\left(\frac{u'}{v'}\right)$ (the friction velocity $v^* \equiv (\sqrt{|\tau|})\underline{z} = z^*$)

$\bar{\underline{p}}^* = \rho f L G \bar{p}^*$

As for the scaling of $\underline{Z}$, two ways are applicable in the present situation.

i) $\underline{z} = \left(\frac{\nu}{f}\right)^{1/2} z$ (smooth laminar scaling)

ii) $\underline{z} = \frac{v^*}{f} z$ (rough turbulent scaling)

Hereafter, either of them will be adopted according to the occasion.
Then Eqs.(2.1) and (2.2) are dimensionalized as

$$R_o\left(\bar{u}\frac{\partial \bar{u}}{\partial r} - \frac{\bar{v}^2}{r}\right) - \bar{v} + \frac{\partial \bar{p}^*}{\partial r} - B\frac{\partial^2 \bar{u}}{\partial z^2} + C\frac{\partial(\overline{u'w'})}{\partial z}$$
$$+ R_o \varepsilon^2\left\{\frac{\partial(\overline{u'^2})}{\partial r} + \frac{1}{r}\frac{\partial}{\partial\theta}(\overline{u'v'}) - \frac{\overline{u'^2+v'^2}}{r}\right\} = 0 \qquad (2.3)$$

$$R_o\left(\bar{u}\frac{\partial \bar{v}}{\partial r} + \frac{\bar{u}\bar{v}}{r}\right) + \bar{u} - B\frac{\partial^2 \bar{v}}{\partial z^2} - E\left(\frac{\partial^2 \bar{v}}{\partial r^2} + \frac{1}{r}\frac{\partial \bar{v}}{\partial r} - \frac{\bar{v}}{r^2}\right) + C\frac{\partial(\overline{v'w'})}{\partial z}$$
$$+ R_o \varepsilon^2\left\{\frac{\partial}{\partial r}(\overline{u'v'}) + \frac{1}{r}\frac{\partial}{\partial\theta}(\overline{v'^2})\right\} = 0 \qquad (2.4)$$

where

$$R_o = \frac{G}{fL},\quad \varepsilon \equiv \frac{v^*}{G},\quad E \equiv \frac{\nu}{fL^2}$$

Coefficients B and C depend on the z-scaling factor.

i) $\underline{z} = \left(\frac{\nu}{f}\right)^{1/2} z$ $B = 1$, $C = R_e \varepsilon^2$ $\left(R_e = \frac{G}{\sqrt{f\nu}}\right)$

ii) $\underline{z} = \frac{v^*}{f} z$ $B = m^2 \equiv \left(\frac{\sqrt{f\nu}}{v^*}\right)^2$, $C = \varepsilon$

In the laboratory experiment by Kreider:

$$R_o \gtrsim \frac{1}{10}, \ \varepsilon \sim \frac{1}{10}, \ E \sim 10^{-6}, \ R_e \sim 10^{2}, \ m \sim \frac{1}{10}$$

$$\left(\frac{\nu}{f}\right)^{1/2} \sim 0.03\ cm, \quad \frac{v^*}{f} \sim 1\ cm$$

Since E and $R_o \varepsilon^2$ are much smaller than other coefficients for both scalings, at least as far as the laboratory experiment is concerned, (2.3) and (2.4) are reduced to

$$R_o\left(\bar{u}\frac{\partial \bar{u}}{\partial r} - \frac{\bar{v}^2}{r}\right) - \bar{v} + \frac{\partial \bar{p}^*}{\partial r} - B\frac{\partial^2 \bar{u}}{\partial z^2} + C\frac{\partial(\overline{u'w'})}{\partial z} = 0 \qquad (2.5)$$

$$R_o\left(\bar{u}\frac{\partial \bar{v}}{\partial r} + \frac{\bar{u}\bar{v}}{r}\right) + \bar{u} - B\frac{\partial^2 \bar{v}}{\partial z^2} + C\frac{\partial(\overline{v'w'})}{\partial z} = 0 \qquad (2.6)$$

Further, nonlinear terms in $\bar{v}$ are linearized in the following way:

(1) Nonlinear terms in (2.5) can be approximated by $-R_o \frac{\bar{v}^2}{r}$ since

$$|\bar{u}| << |\bar{v}|$$

(2) Define geostrophic-cyclostriphically balanced wind speed v_{gc} as

$$-R_o\frac{v_{gc}^2}{r} - \left(v_{gc} - \frac{\partial \bar{p}^*}{\partial r}\right) = 0$$

or replacing $\frac{\partial \bar{p}^*}{\partial r}$ with geostrophic wind v_g, as

$$-R_o\frac{v_{gc}^2}{r} - (v_{gc} - v_g) = 0. \qquad (2.7)$$

(3) Replace v^2 by $v_{gc}\, v$ (Oseen's approximation)

(4) Nonlinear term in (2.6) totally vanishes since

$$\bar{u}\frac{\partial \bar{v}}{\partial r} + \frac{\bar{u}\bar{v}}{r} \sim \bar{u}\left(\frac{\partial v_{gc}}{\partial r} + \frac{v_{gc}}{r}\right) = 0 \quad \text{if} \quad v_{gc}\, r = \text{const.}$$

Then (2.5) and (2.6) become

$$\begin{cases} B\dfrac{d^2 u}{dz^2} + C\dfrac{d(-\overline{u'w'})}{dz} + (1 + R_o')\cdot(v - v_{gc}) = 0 & (2.8) \\[2ex] B\dfrac{d^2 v}{dz^2} + C\dfrac{d(-\overline{v'w'})}{dz} - u = 0 & (2.9) \end{cases}$$

where $R_o' \equiv \frac{v_{gc}}{r} R_o = \frac{v_{gc}}{f r}$, and overbars are dropped from the mean velocity components.

Now that $\frac{\partial}{\partial r}$ no longer appears in the equation, we can regard r as constant and identify it with L, so that $r \equiv 1$. Also v_{gc} is identified with G, so that $v_{gc} \equiv 1$. Then $R_o' = R_o$. (2.10)

III. Modification of Ellison's Theory

In his theoretical study on turbulent Ekman layer, Ellison (1956) assumed

$$\begin{cases} -\overline{u'w'} = \kappa u^* z \dfrac{\partial \bar{u}}{\partial z} \\ -\overline{v'w'} = \kappa u^* z \dfrac{\partial \bar{v}}{\partial z} \end{cases}$$. ($\kappa = 0.41$: von Karman's const.)

When this is applied to our basic equations (2.8) and (2.9) they become

$$\frac{d}{dz}\left(z\frac{du}{dz}\right) + a(1+R_0')(v - v_{gc}) = 0 \tag{3.1}$$

$$\frac{d}{dz}\left(z\frac{dv}{dz}\right) - au = 0 \tag{3.2}$$

where $a \equiv \frac{1}{\kappa \varepsilon R_e}$, when $z = \left(\frac{\nu}{f}\right)^{1/2} z$, and $a \equiv \frac{1}{\kappa}$ when $z = \frac{v^*}{f} z$.

The solution is easily obtained as

$$\frac{u}{\sqrt{1+R_0'}} + i(v - v_{gc}) = A' H_0^{(1)} \cdot \left((4ia'z)^{1/2}\right) \tag{3.3}$$

where $a' \equiv a\sqrt{1+R_0'}$ (3.4). (3.3) is the Ellison's solution when $R_0' = 0$.

$$u + i(v - v_g) = A H_0^{(1)} \left((4iaz)^{1/2}\right). \tag{3.3'}$$

To apply the same boundary conditions as Ellison does, our solution (3.3) should be written in the "stress-wire" coordinate $(\tilde{u}, \tilde{v})$ which turns θ by α , the turning angle of Ekman spiral, so that

$$u = -\tilde{u}\sin\alpha + \tilde{v}\cos\alpha \tag{3.4}$$

$$v = \tilde{u}\cos\alpha + \tilde{v}\sin\alpha \tag{3.5}$$

$$\left(\lim_{z\to 0} \frac{\tilde{v}}{\tilde{u}} \longrightarrow 0\right)$$

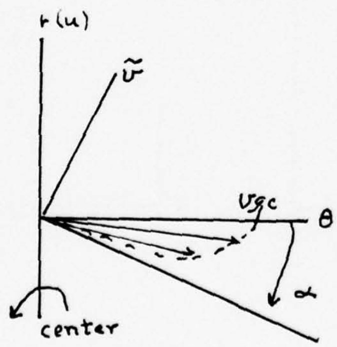


Fig.2

Then, applying Ellison's boundary condition, which is

$$\left.\begin{aligned} \tilde{u} &= \frac{u^*}{\kappa} \ln \frac{z}{z_0} \\ \tilde{v} &= 0 \end{aligned}\right) \quad \text{for small } z, \tag{3.6}$$

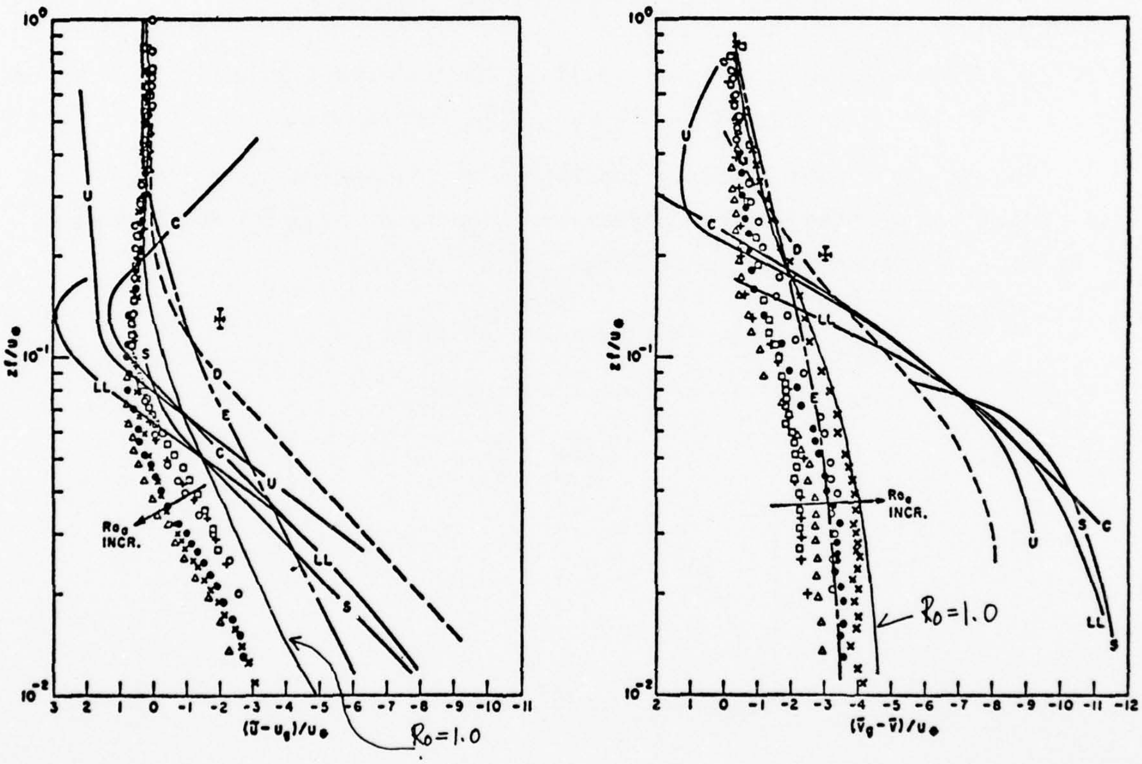
we obtain the solution modified by

$$\begin{cases} \tilde{v} - \tilde{v}_{gc} = P(\eta)(\tilde{v} - \tilde{v}_g)_{\text{Ellis}}, \\ \tilde{u} - \tilde{u}_{gc} = (\tilde{u} - u_g)_{\text{Ellis}} - Q(\eta)(\tilde{v} - \tilde{v}_g)_{\text{Ellis}}, \\ \tan\alpha = \dfrac{\frac{\pi}{2}\sqrt{\eta}}{C}, \end{cases} \tag{3.7}$$

$$\eta \equiv 1 + R_o', \quad C \equiv \ln\left(\frac{\kappa V^*}{4 f z_0}\right) + \underbrace{2(\ln 2 - \gamma)}_{\simeq 0.232}.$$

Result

The modified curves of $(\tilde{v}_g - \tilde{v})/u^*$ and $(\tilde{u} - \tilde{u}_g)/u^*$ for $R_o = 1.0$ is superposed on Kreider's figures. (Fig.3).



It is shown that the modified curves are in qualitative consistency with dependence of the profile obtained by the laboratory data on R_o (R_{oa} in his paper).

In addition (3.7) explains the experimental fact that α increases with R_o. In Ellison's theory, in which $R_o \equiv 0, \alpha$ would rather decrease with the increase of $\frac{\kappa u^*}{4 f z_o}$, which is, in Kreider's data, somewhat associated with the increase of R_{oa}.

Note 1: It should be ascertained, of course, that C increases less rapidly with R_o compared to $\sqrt{\eta}$, for (3.7) to explain the increase of α , but Kreider's data is too sparse to investigate this point.

Note 2: In Kreider's Fig.18 and 19, Ellison's theoretical curves do not seem to coincide with his data of neither $\tilde{u}_g - \tilde{u}$ nor $\tilde{v}_g - \tilde{v}$ even at the limit of $R_{oa} \to 0$.

Although U and V itself in the experimental data should coincide with Ellison's curve since u^* , z_o , and α are determined so that the profile of U should match the logarithmic law at the points closest to the surface ((29) and (30) in Kreider's), there is no necessity that u_g as measured by Kreider should coincide with Ellison's solution for U_G , which is

$$U_G = \frac{u^*}{\kappa}\left(\ln \frac{\kappa u^*}{4 f z_o} + 0.232\right).$$

The discrepancy between them is approximately equal to u^* .

It is one of the demerits in Ellison's theory that the value of geostrophic wind speed is given only by the surface values, regardless of the pressure gradient in the mean field.

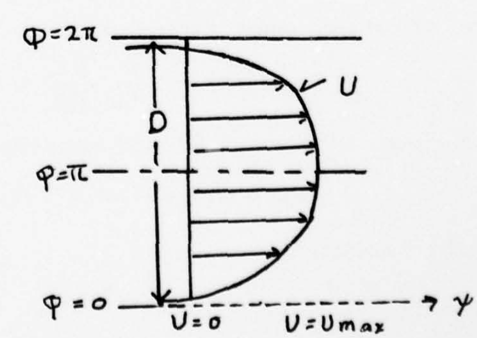
IV <u>Application of Malkus theory to the Ekman Layer.</u>

IV-1. Vertical profile of mean velocity in one-dimensional channel flow.

Malkus (1978) derived the vertical profile of the one-dimensional neutrally stratified steady channel flow velocity with quite a unique method, based on the following two assumptions.

(i) The vertical profile of the mean velocity should have no inflexion point, so that

$$\frac{d^2 U}{d\varphi^2} < 0 \quad (\text{or} > 0) \quad \text{for } 0 < \varphi < 2\pi \qquad (4.1)$$



This condition can be expressed in a Fejer's series as

$$\frac{d^2U}{d\varphi^2} \propto I^* I, \quad I(\varphi) = \sum_{K=0}^{\infty} I_K e^{iK\varphi}$$

(ii) "Spectral Smoothness" - Spectrum I_K is supposed to be smoothed in the sense that $(\Delta I)_K \equiv I_{K+1} - I_K = O(I_0/K_\nu)$ for some K_ν, for which

$$I_K \geq K_\nu \simeq 0. \tag{4.2}$$

These assumptions lead to the result that, for $\varphi >> k_\nu^{-1}$,

$$\frac{d^2U}{d\varphi^2} = \frac{U_\tau}{\pi^2} \frac{|I_0|^2}{4} \operatorname{cosec}^2 \frac{\varphi}{2}, \tag{4.3}$$

where

$$U_\tau^2 \equiv \left(-\frac{1}{\rho}\frac{\partial \bar{P}}{\partial x}\right)\frac{D}{2} = \text{const.} \tag{4.4}$$

Hence

$$\frac{U_{max} - U}{U_\tau} = \frac{|I_0|^2}{\pi^2} \ln\left\{\operatorname{cosec}\left(\frac{\varphi}{2}\right)\right\} \tag{4.5}$$

This velocity profile approaches the logarithmic law in the lower layer but becomes a parabatic profile near the center of the channel. The point in this theory is that the vertical profile of the mean velocity of the neutrally stratified shear flow is determined uniquely without any assumption of "eddy viscosity".

If we suppose, as Ellison did, that the wind profile in (4.9) should match the well-known logarithmic profile in the "matched" or "overlapping" region, which is

$$\underline{U} = \frac{u^*}{\kappa} \ln \frac{z}{z_0} \qquad (\kappa \simeq 0.4 \text{ :von Karman's const.})$$

or, in nondimensionalized form, $U = \frac{1}{\kappa} \ln \varphi + \text{const.}$ (4.6)

then, equating the coefficients of $\ln \varphi$ of (4.9) for $\varphi << 1$ with $\frac{1}{\kappa}$ we get

$$I_0^2 = \pi^2/\kappa, \tag{4.7}$$

which yields

$$U - U_{max} = \frac{U^*}{\kappa} \ln\left(\sin\frac{\varphi}{2}\right) \tag{4.8}$$

The profile of momentum flux is obtained from (4.3) and the basic equation for the steady channel flow, which is written in the dimensional form as

$$-\frac{1}{\rho}\frac{\partial P_0}{\partial x} + \nu \frac{\partial^2 U}{\partial z^2} + \frac{\partial(-\overline{u'w'})}{\partial z} = 0 \tag{4.9}$$

Defining U_τ as (4.4) and nondimensionalizing the variables as

$$\underline{U} = U_\tau U, \quad u'w' = U_\tau^2 u'w', \quad z = \frac{D}{2\pi}\varphi,$$

(4.9) becomes

$$\frac{1}{\pi} + \frac{1}{R'e}\frac{\partial^2 U}{\partial \varphi^2} + \frac{\partial}{\partial \varphi}(-u'w') = 0, \tag{4.10}$$

where $R'_e \equiv \frac{U_\tau D}{2\nu\pi}$ is the stress Reynolds number ($\equiv \frac{R_\tau}{\pi}$ in Malkus, (1978)).

Likewise, (4.3) and (4.4) are nondimensionalized as

$$\frac{d^2U}{d\varphi^2} = -\frac{I_o^2}{4\pi^2}\,\mathrm{cosec}^2\,\frac{\varphi}{2} \tag{4.11}$$

and

$$U - U_{max} = \frac{I_o^2}{\pi^2}\ln\left(\sin\frac{\varphi}{2}\right) \tag{4.12}$$

Thus, from (4.10), (4.11) and (4.7) the divergence of momentum flux is

$$\frac{d}{d\varphi}\left(-u'w'\right) = -\frac{1}{R'_e}\frac{d^2U}{d\varphi^2} - \frac{1}{\pi} = \frac{1}{4\pi R'_e}\,\mathrm{cosec}^2\,\frac{\varphi}{2} - \frac{1}{\pi} \tag{4.13}$$

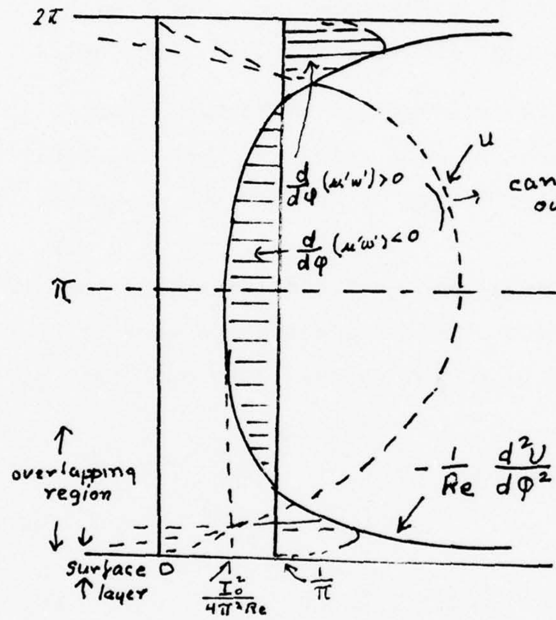


Fig. 4

This relation is illustrated in Fig. (4). The dotted line that branches off the curve for $-\frac{1}{Re}\frac{d^2U}{d\varphi^2}$ denotes what should be expected in the furface layer ($0 < \varphi < \delta - 1$) in which (4.8) is not applicable. The actual boundary conditions for Reynolds stress should be

$$-\overline{u'w'} = \frac{d}{d\varphi}(-u'w') = 0 \quad \text{at } \varphi = 0, 2\pi$$

and hence

$$\int_\delta^{2\pi}\left(\frac{1}{Re}\frac{d^2U}{d\varphi^2} - \frac{1}{\pi}\right)d\varphi = \int_\delta^{2\pi} -\frac{d}{d\varphi}(-u'w')\,d\varphi = 0 \tag{4.14}$$

Since δ, the thickness of the surface layer is expected to be much smaller than π, (4.14) is approximated by

$$\int_\delta^{\pi}\left(\frac{1}{Re}\frac{d^2U}{d\varphi^2} + \frac{1}{\pi}\right)d\varphi = 0,$$

which yields

$$\delta = \frac{1}{\kappa R'_e} \quad \text{and} \quad \left(\frac{-d^2 U}{d\varphi^2}\right)_{max} = (\kappa R_e'^2).$$

Note that δ corresponds to the measure of K_v^{-1} in (4.2)

IV-2 Application to the Ekman flow.

In a first approximate application of these ideas we assume that

(I) The profile of the divergence of momentum flux obtained by Malkus theory (4.13) is applied to both

$$\frac{d}{dz}\left(-\overline{uw}\right) \text{ and } \frac{d}{dz}\left(-\overline{vw}\right)$$

(ii) The direction of the stress does not change with z , regardless of the turn of main flow, so that in the stresswise coordinates

(I) means that the two assumptions in Malkus' theory, (4.1) and (4.2), are valid for each component of mean velocity, U or V. Modification of I_K spectrum due to the Coriolis forces is, of course, expected, but will be left for future study.

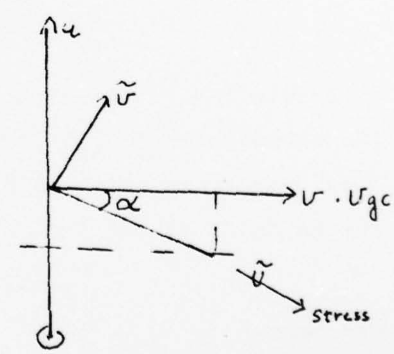
(ii) comes from somewhat physical consideration, that the momentum lux is caused by an intermittent violent penetrative convection which shoots up too quickly to feel the rotation of earth. Then our basic equations (2.8) and (2.9) become

$$\frac{\partial^2 u}{\partial z^2} + (1+R'_o)(v - v_{gc}) = -\varepsilon^2 R_e \frac{\partial(-u'w')}{\partial z} = -\varepsilon^2 R_e f(z)(-\sin\alpha) \qquad (4.15)$$

$$\frac{\partial^2 v}{\partial z^2} - u = -\varepsilon^2 R_e \frac{\partial(-v'w')}{\partial z} = -\varepsilon^2 R_e f(z)\cos\alpha \qquad (4.16)$$

where $f(z) = 2b\left(\frac{1}{4\kappa R'_e}\,\frac{1}{\sin^2 bz} - \frac{1}{\pi}\right)$

$$\begin{cases} b \equiv \dfrac{\pi}{D\left(\frac{\varepsilon}{\nu}\right)^{1/2}} \\ R'_e \equiv \dfrac{v^* D}{2\nu\pi} \end{cases}$$



As we did for the solution of the Modified Ellison's equation, turn the coordinates by α to the stress wise direction.

$$\begin{cases} u = -\tilde{u}\sin\alpha + \tilde{v}\cos\alpha \\ v = \tilde{u}\cos\alpha + \tilde{v}\sin\alpha \end{cases}$$

When $R'_o \neq 0$ it turns out to be quite complicated, but when $R'_o = 0$, (4.13) and (4.14) are transformed to

$$\frac{\partial^2 \tilde{u}}{\partial z^2} - (\tilde{v} \cdot \tilde{v}_{gc}) = -\mathcal{E}^2 Ref(z) \tag{4.17}$$

$$\frac{\partial^2 \tilde{v}}{\partial z^2} + (u - \tilde{u}_{gc}) = 0 \tag{4.18}$$

$$\tilde{v}_{gc} \equiv v_{gc} \sin\alpha, \ \tilde{u}_{gc} \equiv v_{gc} \cos\alpha.$$

Using a complex function $w(z) \equiv (\tilde{u} - \tilde{u}_{gc}) + i(v - v_{gc})$ and (4.18) are united into

$$w'' + iw = -\mathcal{E}^2 Ref(z) \equiv g(z). \tag{4.19}$$

The solution is formally given as

$$\left.\begin{aligned} & w(z) = A(z)e^{\lambda z} + B(z)e^{-\lambda z}. \\ & \text{where } \lambda = \frac{1-i}{\sqrt{2}}, \\ & A(z) \equiv \frac{1}{2\lambda}\int g(z)e^{-\lambda z}dz, \\ & B(z) \equiv -\frac{1}{2\lambda}\int g(z)e^{\lambda z}dz. \end{aligned}\right\} \tag{4.20}$$

Since g(Z) contains cosec^2 bZ, this solution cannot be expressed in tabulated functions. Thus we try to get its approximating solution by an iterative method.

Defining $y \equiv bz$, (4.17) and (4.18) become

$$b^2\tilde{u}'' + (\tilde{v}_{gc} - \tilde{v}) = -a \operatorname{cosec}^2 y + C \tag{4.21}$$

$$b^2\tilde{v}'' - (\tilde{v}_{gc} - \tilde{u}) = 0 \tag{4.22}$$

where

$$a = \mathcal{E}^2 Re\, 2b \frac{1}{4KR'e}, \text{ and } C \equiv \mathcal{E}^2 Re \frac{2b}{\pi}$$

Expand $\tilde{v}(y)$ as $\tilde{v}_0(y) + \delta\tilde{v}_1(y)$, where $\tilde{v}_0(y)$ is defined as

$$(\tilde{v}_{gc} - \tilde{v}_0) \equiv \tilde{v}_{gc}\left(1 - \frac{y}{\pi/2}\right),$$

Then from (4.21)

$$b^2\tilde{u}_0'' = -a \operatorname{cosec}^2 y + C + \frac{\tilde{v}_{gc}}{\frac{\pi}{2}}\left(y - \frac{\pi}{2}\right)$$

$$b^2\tilde{u}_0' = a \operatorname{cosec} y + C\left(y - \frac{\pi}{2}\right) + \frac{v_{gc}}{\pi}\left(y - \frac{\pi}{2}\right)^2$$

$$b^2\tilde{u}_0 = a \ln(\sin y + \frac{C}{2}\left(y - \frac{\pi}{2}\right)^2 + \frac{v_{gc}}{\pi}\frac{1}{3}\left(y - \frac{\pi}{2}\right)^3 + b^2\tilde{u}_{gc}.$$

Thus
$$\tilde{u}_0 = \tilde{u}_{gc} + \frac{a}{b^2}\ln(\sin y) + \frac{1}{b^2}p(y), \tag{4.23}$$

where $p(y) \equiv \frac{v_{gc}}{3\pi}\left(y - \frac{\pi}{2}\right)^2\left\{y - \frac{\pi}{2}\left(1 - \frac{3c}{v_{gc}}\right)\right\}$

Instead of going to further iteration, it would be more helpful to try to

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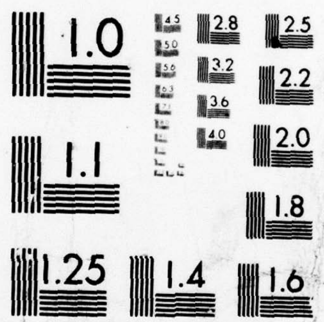

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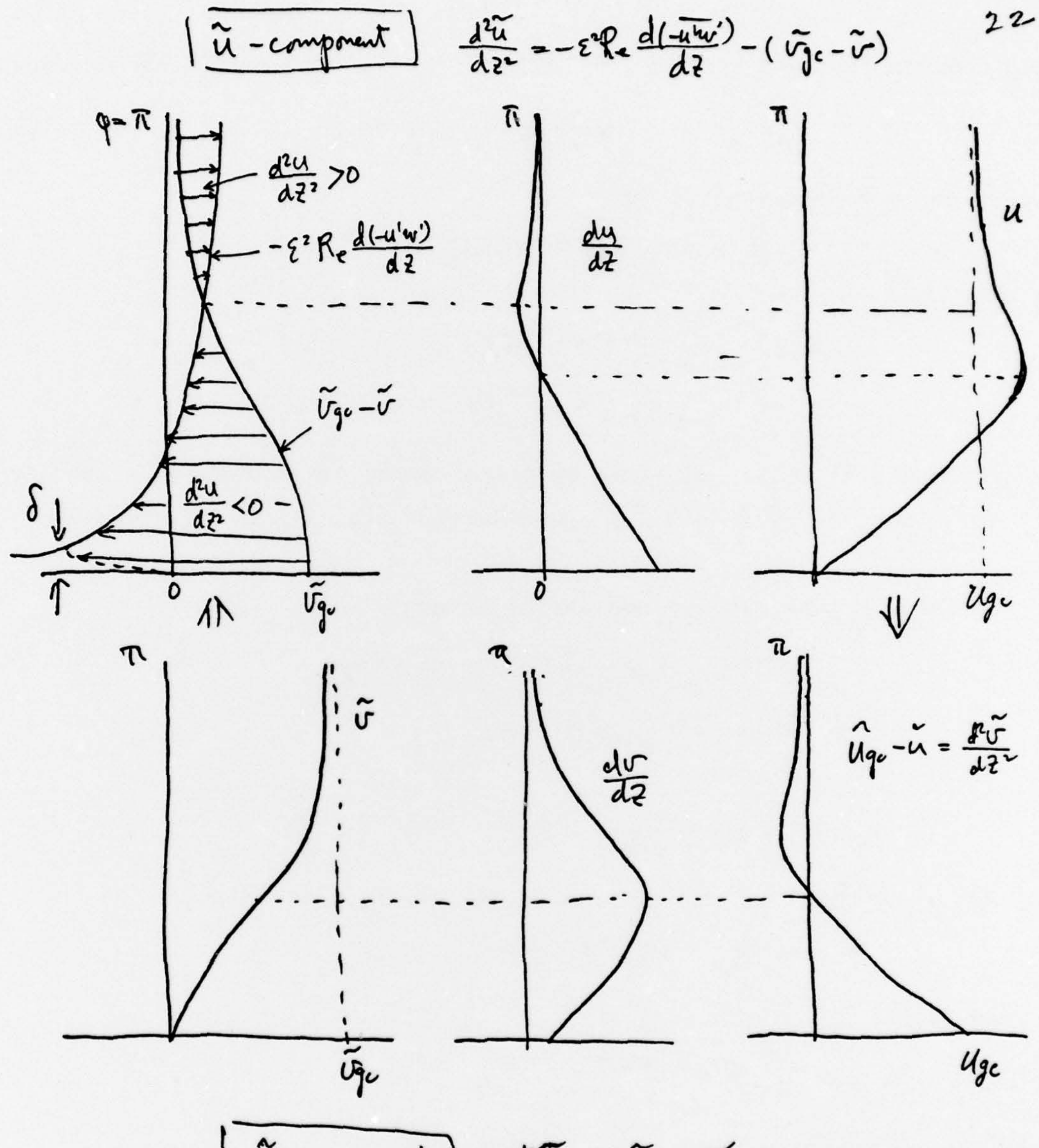
obtain a consistent set of rough profiles for $\tilde{u}$ and $\tilde{v}$ satisfying (4.17) and (4.18) by illustrating the curves of u , v , u'' and v'' with the aid of u_0 in (4.23).



Rough illustration of curves satisfying (4.17) and (4.18)

It is shown in this rough illustration that $\tilde{u}$ profile overshoots u_{gc}, as in Ellison's curve and in Kreider's experimental curve, and that the profiles for both $\tilde{u}$ and $\tilde{v}$ have one inflexion point, in contrast to the oscillating behavior of Ellison's solution.

Acknowledgment

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References

Caldwell, D.R., C.W.Van Atta and K.N.Helland 1972 A laboratory study of the turbulent Ekman Layer. Geophys.Fluid Dynamics 3: 125-160.

Ellison, T.H. 1956 Atmospheric turbulence. G.I.Taylor Anniversary Volume (Cambridge Univ.Press).

Kreider, J. 1973 A laboratory study of the turbulent Ekman Layer. Dissertation.

Malkus, W.V.R. 1978 Turbulent velocity profiles from stability criteria. J.Fluid Mech. (in press).

STEADY BUOYANT PLUMES IN A FLUID OF LARGE PRANDTL NUMBER AND TEMPERATURE DEPENDENT VISCOSITY

Dean S. Oliver

1. Introduction

Quite often in fluid dynamics the assumption of large Prandtl number, $\sigma = \frac{\nu}{\kappa}$, will simplify a problem. This is certainly the case in Bénard convection where it allows the inertial terms in the momentum equation to be neglected. Large Prandtl number fluids also seem to be more stable to transitions to time-dependent motions. Yet by changing the problem only slightly to consideration of laminar convection from either a line source or a point source the infinite Prandtl number assumption actually makes the solution more difficult. Closed solutions are well-known for laminar plumes in fluids of Prandtl number equal to 1 or 2 (Fujii, 1963), but in the limit of large constant Prandtl number the equations have only been solved numerically and even then the solution is much different from the $\sigma = 0$ (1) solutions.

Still, it is the large Prandtl number case that we expect to be applicable in the earth's mantle and further we believe that the viscosity of the mantle material is strongly temperature dependent. So whether the interest is in fast viscous convection applied to the earth (Roberts, 1977) or in the formation of island chains from discrete plumes in the mantle (Skilbeck, Whiteheat,

1978) the problem of buoyant plumes in a fluid of large Prandtl number and temperature dependent viscosity should be important.

2. <u>The Plume Equations</u>

In order to study the plume formed above a line source of heat in an infinite fluid we use the coordinates shown below. w is the vertical or z-component of velocity defined by the direction of the gravitational acceleration and u is the horizontal velocity component.

In general, the viscosity of the fluid will be allowed to be a function of the temperature.

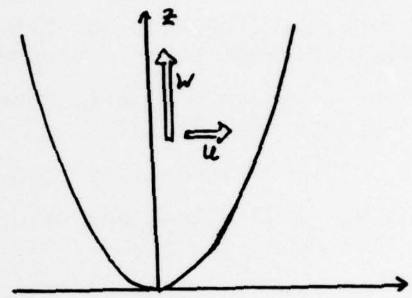

Fig. 1

The equations for the velocity and temperature fields in this problem are:

$$u\frac{\partial w}{\partial x} + w\frac{\partial u}{\partial z} = \alpha g\theta + \nu\frac{\partial^2 w}{\partial x^2} + \frac{\partial \nu}{\partial \theta}\frac{\partial \theta}{\partial x}\frac{\partial w}{\partial x}$$

$$u\frac{\partial \theta}{\partial x} + w\frac{\partial \theta}{\partial z} = \kappa\frac{\partial^2 \theta}{\partial x}$$

$$\frac{\partial u}{\partial x} + \frac{\partial w}{\partial z} = 0$$

and

$$\int_{-\infty}^{\infty} w\theta\, dx = F_0$$

where Eq.(2.4) states that the vertical heat flux must be constant at all heights in the plume. The fluid has been assumed to be Boussinesq (except for the variation in viscosity) and the boundary layer approximation has been made whereby pressure forces and vertical derivatives are negligably small compared to horizontal gradients. The conditions justifying the boundary layer approximation are examined later for each case individually.

It is possible to obtain similarity solutions for the temperature and the velocity when the viscosity is proportional to a power of the temperature perturbation. A plume in a fluid of constant viscosity is considered first because it can be compared to results obtained by others (Spalding, Cruddace, 1961). The case of viscosity inversely proportional to the temperature is then examined

because a solution has been found in the large Prandtl number limit.

3. The Constant Viscosity Plume

If we introduce a stream function, ψ, such that $u = \frac{\partial \psi}{\partial z}$ and $w = \frac{-\partial \psi}{\partial x}$ then Eq.(2.1), (2.2) and (2.4) become

$$-\frac{\partial \psi}{\partial z}\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial \psi}{\partial x}\frac{\partial^2 \psi}{\partial x \partial z} = \alpha g \theta - \nu \frac{\partial^3 \psi}{\partial x^3} \tag{3.1}$$

$$\frac{\partial \psi}{\partial z}\frac{\partial \theta}{\partial x} - \frac{\partial \psi}{\partial x}\frac{\partial \theta}{\partial z} = K \frac{\partial^2 \theta}{\partial x^2} \tag{3.2}$$

$$\int_{-\infty}^{\infty} \theta \frac{\partial \psi}{\partial x} dx = -F_o \quad . \tag{3.3}$$

It is quite easy to nondimensionalize these equations such that the inertia terms are unimportant and the boundary layer approximation is satisfied. To do this we introduce a Grashof number, $G = \frac{F_o \alpha g z^3}{K^2 \nu}$, as well as nondimensional similarity functions, Ψ and Θ, such that

$$\psi(x,z) = K G^{1/5} \Psi(\xi)$$

and

$$\theta(x,z) = \frac{F_o}{K} G^{-1/5} \Theta(\xi)$$

where $\xi = \frac{x}{z} G^{1/5}$ is the similarity variable. Substitution of these functions into (3.1), (3.2) and (3.3) leads to a set of equations which are identical to the equations of Spalding (1961) except that the Prandtl number multiplies different terms.

$$\frac{K}{\nu}\left[\frac{1}{5}\left(\frac{d\Psi}{d\xi}\right)^2 - \frac{3}{5}\Psi\frac{d^2\Psi}{d\xi^2}\right] = \Theta - \frac{d^3\Psi}{d\xi^3} \tag{3.4}$$

$$\frac{d}{d\xi}\left(\Theta\Psi\right) = \frac{5}{3}\frac{d^2\Theta}{d\xi^2} \tag{3.5}$$

$$\int_{-\infty}^{\infty} \Theta \frac{d\Psi}{d\xi} = -1 \, . \tag{3.6}$$

It is now easy to see that the condition under which the boundary layer approximation is justified is that the Grashof number be much greater than 1. We can proceed further by integrating (3.5) and substituting the solution for Ψ into (3.4).

$$\frac{K}{\nu}\left[\frac{25}{3}\left(\frac{d^2}{d\xi^2}\ln\Theta\right)^2 - \frac{5}{3}\left(\frac{d}{d\xi}\ln\Theta\right)\left(\frac{d^3}{d\xi^3}\ln\Theta\right)\right] = \Theta - \frac{5}{3}\frac{d^4}{d\xi^4}\ln\Theta \tag{3.7}$$

Closed solutions to this equation have been found for the special cases when the Prandtl number, σ, is equal to 5/9 and for $\sigma = 2$ (Yig, 1953). Numerical solutions for $\sigma = 0.01$, 0.7, and 10 have been obtained by Fujii (1962) and the infinite Prandtl number limit was considered by Spalding and

Cruddace (1961). It is the large Prandtl number case which we wish to examine here.

We define new variables

$$\varphi = -\ln \Theta / \Theta(0) \quad \text{and} \quad \xi = \left(\frac{5}{3\Theta(0)}\right)^{1/4} \eta .$$

Then Eq.(3.7) becomes

$$\frac{1}{\sigma}\left[5\varphi''^2 - \varphi'\varphi'''\right] = e^{-\varphi} + \varphi'''' \tag{3.8}$$

It is straightforward to show that if φ is expanded in a power series of an unknown small parameter ε such that $\varphi = \varepsilon\varphi_1 + \varepsilon^2\varphi_2 + \ldots$, the solution satisfying the boundary conditions is

$$\varphi = \frac{a\eta^2}{2} - \frac{\eta^4}{4!}\left(1 - \frac{5a^2}{\sigma}\right) + O(\eta^6)$$

where ε has now been absorbed into the constant a. However, near the origin where $X \ll 1$ we can presumably neglect the $O(\eta^4)$ term also. To lowest order the constant a is then determined to be $a = \sqrt{\frac{\sigma}{5}}$. If the Prandtl number is large, this approximate solution is quite good and the temperature drops to nearly zero a short distance from the plume axis. The lowest order approximation is

$$\Theta = \Theta_0 \exp\left(-\frac{1}{2}\sqrt{\frac{\sigma}{5}}\,\eta^2\right)$$

$$= \Theta_0 \exp\left(-\frac{\sqrt{3\sigma\Theta_0}}{10}\,\xi^2\right)$$

This must satisfy the integral condition (3.6). If the Prandtl number is very large than the approximate solution can be used when evaluating the integral and the result is

$$\Theta = \left(\frac{27}{100\pi^2\sigma}\right)^{1/5}$$

Putting all the dimensional dependence back into the solution

$$\Theta(x,z) = \left(\frac{27 F_0^4}{z^3 K^2 \alpha g\, 100\pi^2 F_0}\right)^{1/5} \exp\left[-\frac{x^2}{10}\left(\frac{F_0^2\alpha^2 g^2\, 81}{10\pi K^4 z^4}\right)^{1/5}\right]$$

which is completely independent of viscosity.

Spalding and Cruddace (1961) arrived at essentially the same approximate solution for the temperature, but they proceeded further and determined the velocity profile for the infinite Prandtl number limit. They also concluded that the results were independent of whether or not the viscosity depends on temperature.

4. The Effect of Temperature Dependent Viscosity

The laminar plume for a fluid of large Prandtl number and constant

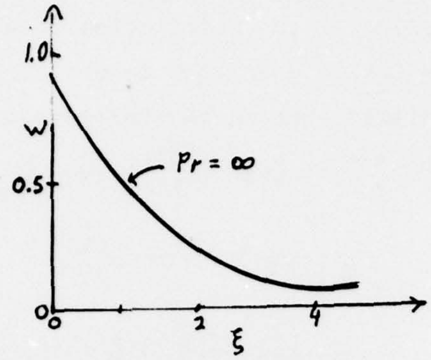

Fig.2. Nondimensional velocity distribution about a line source from Cruddace (1961).

viscosity is characterized by two very different length scales. A thermal boundary layer whose nondimensional width is of the order of the inverse Prandtl number forces motions in a plume of unit nondimensional width. Spalding and Cradduce incorrectly reasoned that since the thermal plume is very thin the same results hold even when the viscosity depends strongly on temperature. By choosing a specific temperature dependence for which similarity solutions can be found, the general effect of variable viscosity can be determined. If we specify a temperature dependence of the form $\nu = \nu_0\left(\frac{\theta}{\theta_0}\right)^{-N}$ then the momentum, heat and continuity equations are

$$u\frac{\partial w}{\partial x} + w\frac{\partial w}{\partial z} = \alpha g\theta + \nu_0\left(\frac{\theta}{\theta_0}\right)^{-N}\frac{\partial^2 w}{\partial x^2} - \frac{N\nu_0}{\theta_0}\left(\frac{\theta}{\theta_0}\right)^{-N-1}\frac{\partial\theta}{\partial x}\ \frac{\partial w}{\partial x} \qquad (4.1)$$

$$u\frac{\partial\theta}{\partial x} + w\frac{\partial\theta}{\partial z} = K\frac{\partial^2\theta}{\partial x^2} \qquad (4.2)$$

$$\frac{\partial u}{\partial x} + \frac{\partial w}{\partial z} = 0 \qquad (4.3)$$

$$\int_{-\infty}^{\infty}\theta w\,dx = F_0 \qquad (4.4)$$

Anticipating that for large N the last term in (4.1) may be larger than the other viscous term, we choose a nondimensionalization which puts a factor of 1/N in front of the viscous dissipation and the inertial terms but makes no approximation in the equations. These scalings give the following substitutions

$$\theta = \theta_0\,\mathcal{G}^{-\frac{1}{5+N}}\,\Theta(\xi)$$

$$\psi = \frac{F_0}{\theta_0}\,\mathcal{G}^{\frac{1}{5+N}}\,f(\xi)$$

where the similarity variable, $\xi = \frac{F_0}{K\theta_0}\mathcal{G}^{\frac{1}{5+1}}\frac{x}{z}$ and $\mathcal{G} \equiv \frac{\alpha g K^3 z^3\theta_0^5}{N\nu_0 F_0^4}$.

The stream function, Ψ, is introduced into Eqs.(4.1), (4.2) and (4.4) to eliminate the continuity equation, then substitution of the similarity variable, , reduces the partial differential equations to ordinary differential equations which, when nondimensionalized, became the following:

$$\left(\frac{F_0^4}{\alpha g K^2 \Theta_0 \bar{z}^3}\right)^{\frac{N}{5+N}} \left(\frac{1}{N\sigma}\right)^{\frac{5}{5+N}} \left[\frac{1}{5}\left(\frac{d\Psi}{d\xi}\right)^2 - \frac{3}{5}\Psi\frac{d^2\Psi}{d\xi^2}\right] = \Theta - \frac{1}{N}\frac{d}{d\xi}\left[\Theta^{-N}\frac{d^2\Psi}{d\xi^2}\right] \quad (4.5)$$

$$\frac{d}{d\xi}\left(\Theta\Psi\right) = \frac{5+N}{3}\frac{d^2\Theta}{d\xi^2} \quad (4.6)$$

$$\int_{-\infty}^{\infty} \Theta \frac{d\Psi}{d\xi} d\xi = -1 \quad (4.7)$$

It might appear hopeless to expect to find a simple solution to this system of equations but a solution has been found for the case where the Prandtl number, $\sigma = \frac{\nu_0}{K}$, is large enough that the inertial terms can be neglected. In the case of N = 1 the equations simplify to

$$\Theta = \frac{d}{d\xi}\left(\frac{1}{\Theta}\frac{d^2\Psi}{d\xi^2}\right) \quad (4.8)$$

$$\Theta\Psi = 2\frac{d\Theta}{d\xi} \quad (4.9)$$

The solution of these equations corresponding to the appropriate boundary conditions is

$$\Theta = 2\sqrt{2}\, a^2 \operatorname{sech}^2 a\xi$$

and

$$\Psi = -4a \tanh a\xi .$$

Finally, the integral condition (4.7) must be satisfied in order to determine the constant a.

$$\int_{-\infty}^{\infty} \Theta \frac{d\Psi}{d\xi} d\xi = -8\sqrt{2}\, a^4 \int_{-\infty}^{\infty} \operatorname{sech}^4 a\xi \, d\xi = -1$$

so $\quad a = \left(\frac{3}{32\sqrt{2}}\right)^{1/3}$

When the temperature, velocity, and viscosity profiles are plotted together on one graph as in Fig.3 it is easy to see why the flow region is only as wide as the thermal layer. The central core of the plume has a relatively constant viscosity which increases rapidly outside the hot thermal region. Naturally, this is the region where the motion must be concentrated.

Some interesting observations about the shape of the plume can be made when the solutions are written in dimensional form.

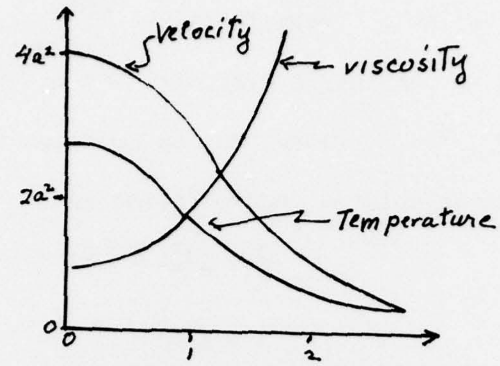

Fig.3 The nondimensional plume quantities for a fluid with viscosity inversely proportional to temperature.

$$\theta(x,z) = 2\sqrt{2}\,a^2\left(\frac{\alpha g K^3}{\nu_0 F_0^4 \theta_0}\right)^{-1/6}\frac{1}{\sqrt{z}}\,\mathrm{sech}^2\left[a\left(\frac{F_0^2\alpha g}{\nu_0 K^3\theta_0}\right)^{1/6}\frac{x}{\sqrt{z}}\right]$$

$$w(x,z) = 4a^2\left(\frac{F_0^2\alpha g}{\theta_0\nu_0}\right)^{1/3}\mathrm{sech}^2\left[a\left(\frac{F_0^2\alpha g}{\nu_0 K^3\theta_0}\right)^{1/6}\frac{x}{\sqrt{z}}\right]$$

$$u(x,z) = -2\left(\frac{F_0^2\alpha g K^3}{\theta_0\nu_0}\right)^{1/6}\frac{a}{\sqrt{z}}\tanh\left[\qquad\qquad\right]$$

$$+2a^2\left(\frac{F_0^2\alpha g}{\theta_0\nu_0}\right)^{1/3}\frac{x}{z}\,\mathrm{sech}^2\left[\qquad\qquad\right]$$

First, notice that the scale width for both components of velocity and the temperature are the same. The envelope of the plume increases in width as the square root of the height but if $\frac{z^3 F_0^2\alpha g}{\nu_0 K^3\theta_0} >> 1$ the plume is widening very slowly as the height increases (and the boundary layer approximation is justified). While the width of the vertical velocity profile is inversely proportional to $\sqrt{K}$, the peak amplitude is independent of thermal diffusivity The peak value of the temperature increases as $K^{-1/2}$ so if the diffusivity is smaller holding all other parameters constant the width of the plume will shrink and the peak temperature at any height will increase.

5. The Axisymmetric Similarity Equations

If we consider a point source of heat instead of a line source the equations are simpler when written in cylindrical polar coordinates. The steady, asymmetric Boussinesq equations in the boundary layer approximation are

$$w\frac{\partial w}{\partial z} + u\frac{\partial w}{\partial x} = \frac{1}{r}\frac{\partial}{\partial r}\left(r\nu\frac{\partial w}{\partial r}\right) + \alpha g\theta \tag{5.1}$$

$$w\frac{\partial\theta}{\partial z} + u\frac{\partial\theta}{\partial r} = \frac{K}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\theta}{\partial r}\right) \tag{5.2}$$

$$\frac{\partial}{\partial z}(rw) + \frac{\partial}{\partial r}(ru) = 0 \tag{5.3}$$

Naturally, we define a stream function, ψ , such that $u = -\frac{1}{r}\frac{\partial\psi}{\partial z}$ and $w = \frac{1}{r}\frac{\partial\psi}{\partial r}$. Then if we consider a viscosity $\nu = \nu_0 \left(\frac{\theta}{\theta_0}\right)^{-N}$ and a vertical heat flux $F_0 = \int_{-\infty}^{\infty} \theta \frac{\partial\psi}{\partial r}\, dr$, the variables can be nondimensionalized and reduced to functions of only one variable in the following way:

$$\theta = \frac{F_0}{z K} \Theta(\xi)$$

$$\psi = K z \Psi(\xi)$$

where
$$\xi = \frac{\alpha g F_0}{K^2 z^2 \nu_0} \left(\frac{F_0}{z K \theta_0}\right)^{N/4}$$

In the limit $\left(\frac{F_0}{z\,\theta_0 K}\right)^N \frac{K}{\nu_0} \longrightarrow \infty$ the axisymmetric equations can be reduced to the nondimensional similarity equations given here.

$$\frac{d}{d\xi}\left[\xi\, \Theta^{-N} \frac{d}{d\xi}\left(\frac{1}{\xi}\,\frac{d\Psi}{d\xi}\right)\right] + \xi\,\Theta = 0 \qquad (5.4)$$

$$\Theta\Psi + \xi\,\frac{d\Theta}{d\xi} = 0. \qquad (5.5)$$

6. The Experiment

We decided to try a very quick experiment to produce a plume in a fluid whose temperature changed markedly with temperature. A sugar water solution was chosen because of the availability of sugar. Approximately 20 lbs. of sugar was mixed with enough water to make three gallons of solution. The specific gravity of the solution was 1.259 at 24.2°C. A rough approximation for the temperature dependence of viscosity for this solution near 25°C was

$$\mu = 20 - .8\,(T - 25) \qquad \text{centistokes}$$

where T is the temperature.

The heater, made by winding nichrome wire around a block, was 10 cm long and 0.5 cm wide and had a resistance of 37 ohms. It was placed near the bottom of a plexiglass tank and the output was adjusted to nearly 70 watts which corresponds to $F_0 = 1.2$ cm^2 °C sec^{-1}.

Figures 4 and 5 are shadowgraph pictures of the plume 15 and 30 seconds after the heater was turned on.. The dark core of the plume is fluid which is less dense and therefore has a lower index of refraction. Figure 6 represents an attempt to determine the velocity distribution across the plume 15 seconds after heating began. Only qualitative conclusions can be drawn from the experiment. Obviously, it is possible to get narrow, almost vertical plumes in a

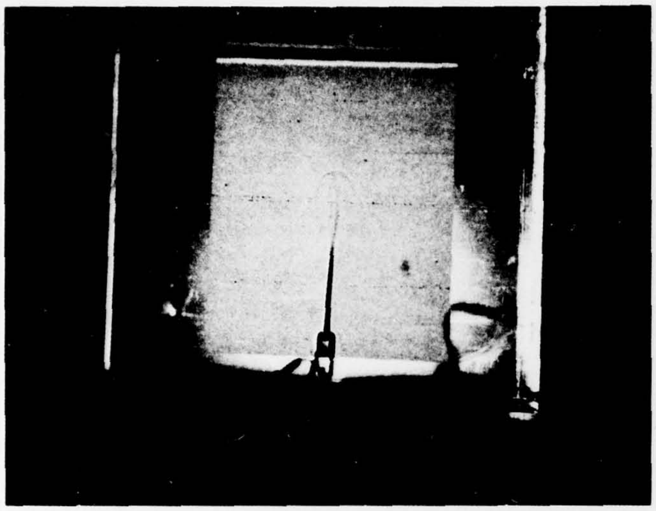

Fig.4 Shadowgraph view of the temperature profile in a developing plume 15 seconds after heating began.

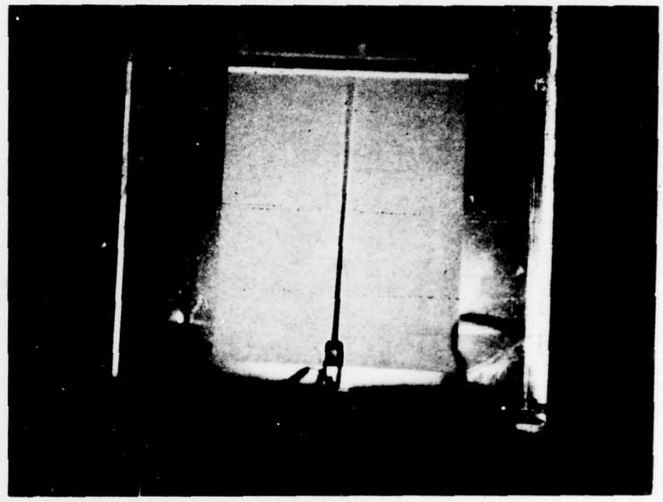

Fig.5 Fully developed plume at 30 seconds after beginning heating.

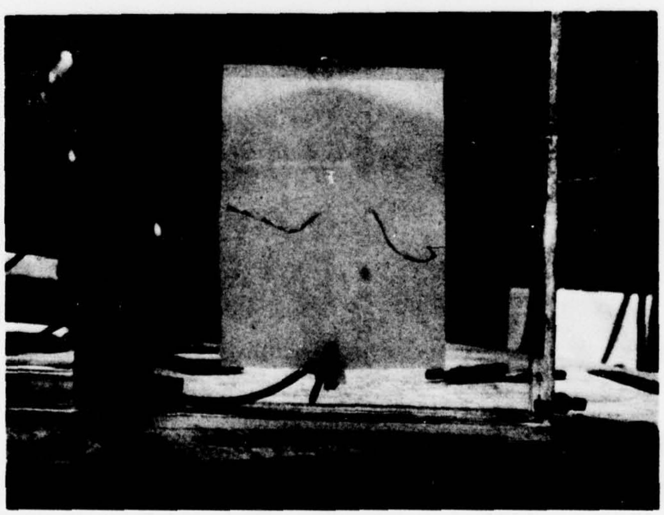

Fig.6 Hypodermic needle drawn through the plume releasing ink to determine the velocity profile. 15 seconds after turning on heater.

fluid with even a slightly temperature dependent viscosity as long as the suitably defined Grashof number is large enough. The velocity distribution shown by the ink line in Fig.6 might be far from the actual steady state distribution because at 15 seconds the plume had still not reached the surface.

7. Conclusions and Remarks

The real hope in studying this problem was that some progress could be made in understanding the structure of plumes which might occur in the mantle. The ages of island chains, the regularity of spacing, the direction of propagation and the chemical composition all suggest that the source of the material is fixed fairly deep in the mantle. If the plume is to have a localized surface expression then there must be some mechanism which keeps the plume from diffusing greatly as it rises. One explanation is that the upwelling material is chemically different from the surrounding material, in which case we expect the types of plumes studied by Whitehead and Luther (1975) to be applicable. Another possibility however is that the plume materials are chemically identical but that the viscosity of the plume is lower because of the higher temperature in that region. The real case is probably a combination of these two possibilities, each approximately valid for different depths.

I might suggest that there are many problems associated with these plumes that remain to be solved. The instability caused by a shear layer above the source directly applies to the geophysical problem of hot spots. And even though the difference may not be great it can be argued that the temperature dependence chosen in this study is not realistic for geophysical fluids. Finally, I suggest that this could be a first step towards determining the asymptotic heat flow behavior at large Rayleigh number in a fluid of variable viscosity.

Acknowledgments

I would like to thank Andrew Soward, John Whitehead and Willem Malkus for some helpful conversations and the guidance they gave.

References

Fujii, T. 1963 Theory of the steady laminar natural convection above a horizontal line heat source and a point heat source, Intern.J.Heat Mass. Transf. 6: 597-606.

Roberts, G.O. 1977 Fast viscous convection, Geophys.Astrophys.Fluid Dymamics, 8: ;97-233.

Skilbeck, J.N. and J.A.Whitehead 1978 Formation of discrete islands in linear island chains, Nature, 272(5653): 499-501.

Spalding, D.B. and R.G.Cruddace 1961 Theory of the steady laminar buoyant flow above a line heat source in a fluid of large Prandtl number and temperature dependent viscosity, Intern.J.Heat Mass.Transf. 3: 55-59.

Whitehead, J.A. and D.S.Luther 1975 Dynamics of laboratory diapir and plume models, J.Geophys.Res., 80: 705-717.

Yih, C.-S. Free convection due to boundary sources, in "Symposium on the use of models in geophysical fluid mechanics", held at The Johns Hopkins University.